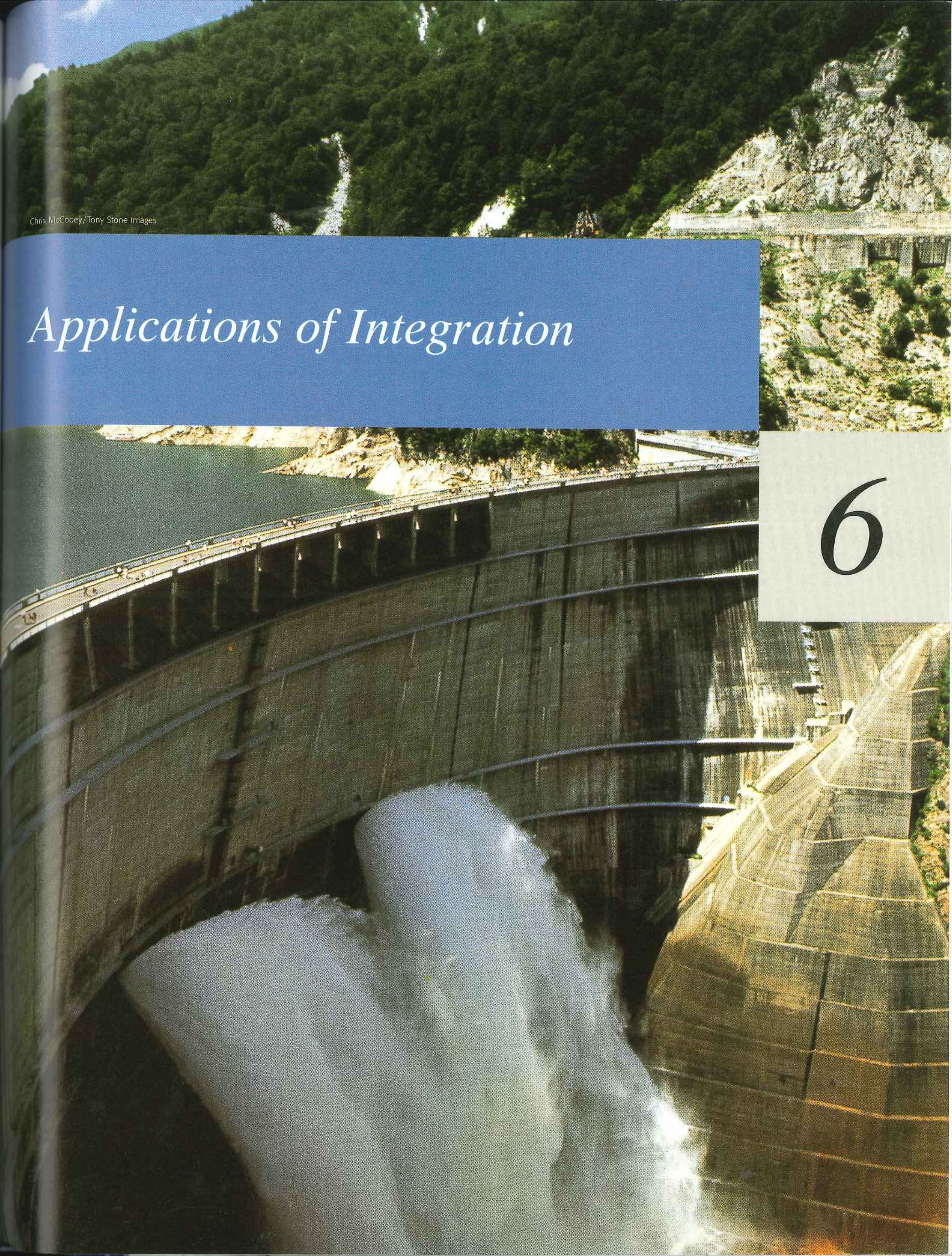
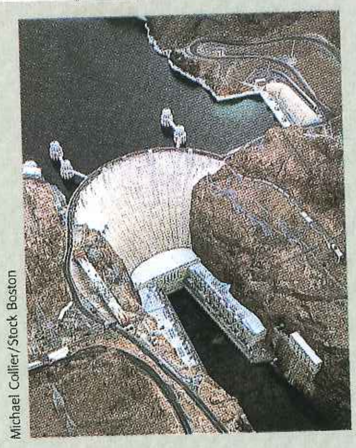


Chris McCooey/Tony Stone Images

Applications of Integration

6

MOTIVATING THE CHAPTER Constructing an Arch Dam



Michael Collier/Stock Boston

HOOVER DAM

Hoover Dam, one of the highest concrete dams in the world, uses a gravity-arch construction. It relies on both the walls of the Black Canyon and its own mass to hold back the waters of the Colorado River.

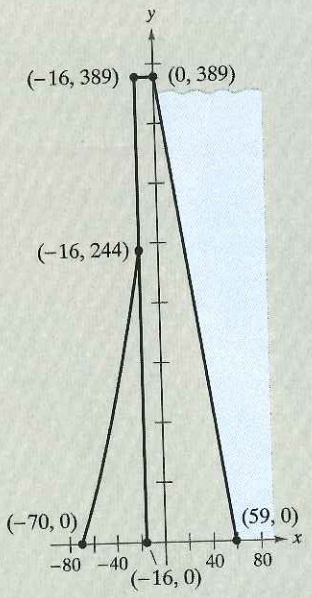
Dams were originally built to ensure water supplies during dry seasons. As technical knowledge has increased, they have begun serving other functions. Today, dams may be built to create recreational lakes, to power generators, and to prevent flooding. Every new dam creates concerns. Along with its benefits, the dam may upset an area's ecology and force the relocation of people and wildlife. Also, a poorly constructed dam endangers the entire surrounding region, creating the possibility of a massive disaster.

There are several designs used in dam construction, one of which is the arch dam. This design curves toward the water it contains, and is usually built in narrow canyons. The force of the water presses the edges of the dam against the walls of the canyon, so the natural rock helps support the structure. This added support means that the arch dam can be built with less construction materials than its gravity-supported counterpart.

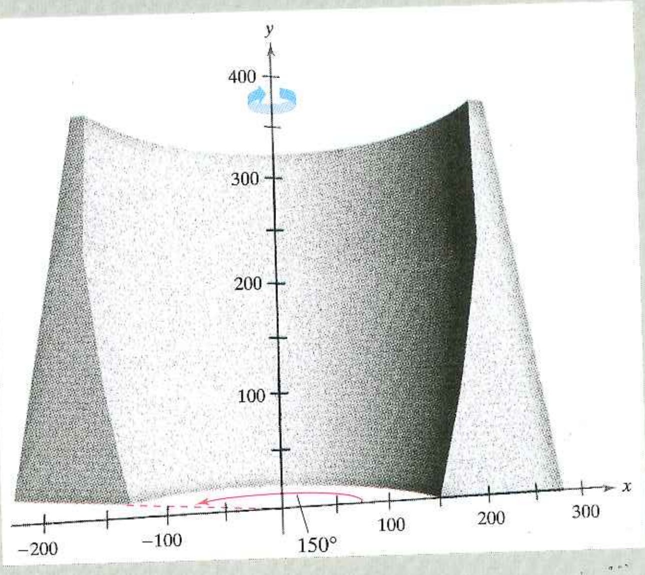
A cross section of a typical arch dam can be modeled as shown in the figure at the lower left. The model for this cross section is as follows.

$$f(x) = \begin{cases} 0.03x^2 + 7.1x + 350, & -70 \le x \le -16 \\ 389, & -16 < x < 0 \\ -6.593x + 389, & 0 \le x \le 59 \end{cases}$$

To form the arch dam, this cross section is swung through an arc, rotating it about the y-axis. The number of degrees through which it is rotated and the length of the axis of rotation vary, depending primarily on how much the water level varies. A possible configuration shows a rotation of 150° and an axis of rotation of 150 feet.



Cross section of an arch dam



FOR FURTHER INFORMATION For more information on the calculus of dam design, see *Calculus, Understanding Change,* a three-part, half-hour video produced by COMAP and funded by the National Science Foundation.

QUESTIONS

1. Find the area of a cross section of the dam.
2. Describe a strategy for estimating the volume of concrete that would be needed to build this dam.
3. Use the strategy to estimate the volume of concrete needed to build the dam described on this page.

The concepts presented here will be explored further in this chapter. For an extension of this application, see the lab series that accompanies this text.

SECTION 6.1 Area of a Region Between Two Curves

Area of a Region Between Two Curves •
Area of a Region Between Intersecting Curves

Area of a Region Between Two Curves

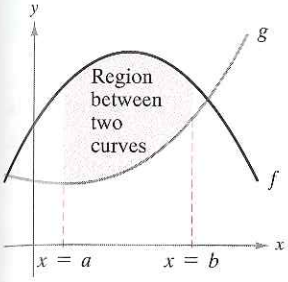
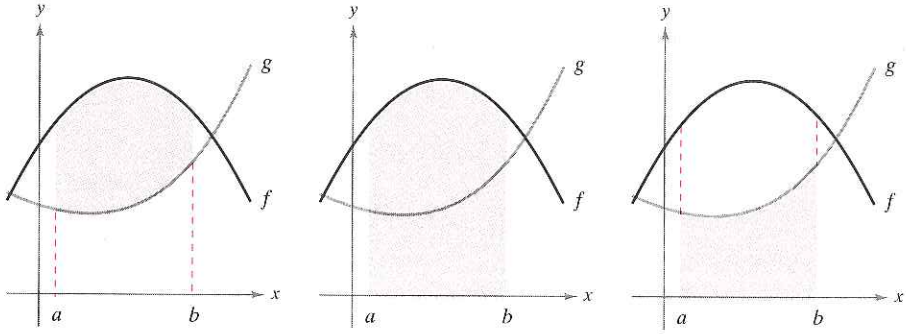


Figure 6.1

With a few modifications you can extend the application of definite integrals from the area of a region *under* a curve to the area of a region *between* two curves. Consider two functions f and g that are continuous on the interval $[a, b]$. If, as in Figure 6.1, the graphs of both f and g lie above the x-axis, and the graph of g lies below the graph of f, you can geometrically interpret the area of the region between the graphs as the area of the region under the graph of g subtracted from the area of the region under the graph of f, as shown in Figure 6.2.



Area of region between f and g	=	Area of region under f	−	Area of region under g
$\int_a^b [f(x) - g(x)]\,dx$	=	$\int_a^b f(x)\,dx$	−	$\int_a^b g(x)\,dx$

Figure 6.2

To verify the reasonableness of the result shown in Figure 6.2, you can partition the interval $[a, b]$ into n subintervals, each of width Δx. Then, as shown in Figure 6.3, sketch a **representative rectangle** of width Δx and height $f(x_i) - g(x_i)$, where x_i is in the ith interval. The area of this representative rectangle is

$$\Delta A_i = (\text{height})(\text{width}) = [f(x_i) - g(x_i)]\,\Delta x.$$

By adding the areas of the n rectangles and taking the limit as $\|\Delta\| \to 0$ $(n \to \infty)$, you obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\,\Delta x.$$

Because f and g are continuous on $[a, b]$, $f - g$ is also continuous on $[a, b]$ and the limit exists. Therefore, the area of the given region is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\,\Delta x$$
$$= \int_a^b [f(x) - g(x)]\,dx.$$

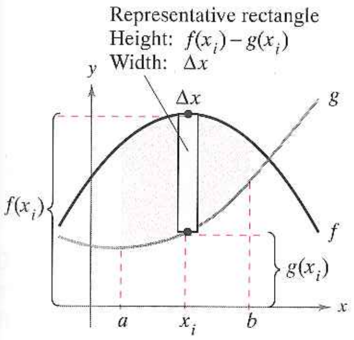


Figure 6.3

SECTION 6.1 Area of a Region Between Two Curves

Area of a Region Between Two Curves •
Area of a Region Between Intersecting Curves

Area of a Region Between Two Curves

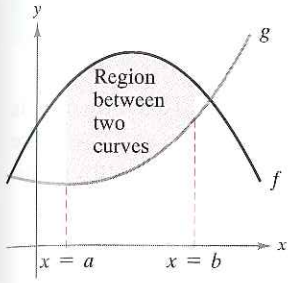
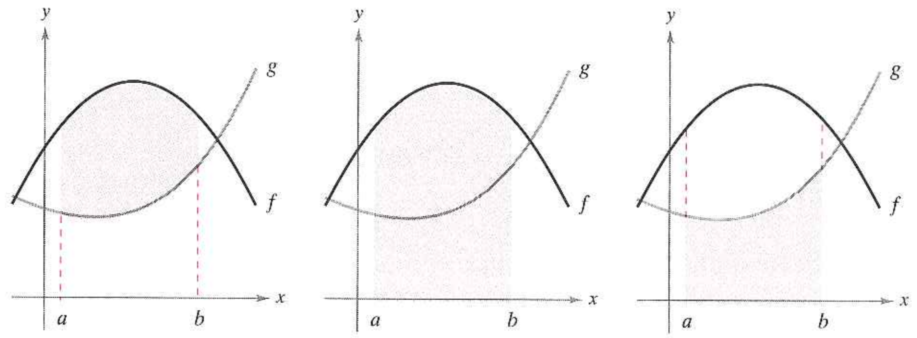


Figure 6.1

With a few modifications you can extend the application of definite integrals from the area of a region *under* a curve to the area of a region *between* two curves. Consider two functions f and g that are continuous on the interval $[a, b]$. If, as in Figure 6.1, the graphs of both f and g lie above the x-axis, and the graph of g lies below the graph of f, you can geometrically interpret the area of the region between the graphs as the area of the region under the graph of g subtracted from the area of the region under the graph of f, as shown in Figure 6.2.



Area of region between f and g	=	Area of region under f	−	Area of region under g

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Figure 6.2

To verify the reasonableness of the result shown in Figure 6.2, you can partition the interval $[a, b]$ into n subintervals, each of width Δx. Then, as shown in Figure 6.3, sketch a **representative rectangle** of width Δx and height $f(x_i) - g(x_i)$, where x_i is in the ith interval. The area of this representative rectangle is

$$\Delta A_i = (\text{height})(\text{width}) = [f(x_i) - g(x_i)]\,\Delta x.$$

By adding the areas of the n rectangles and taking the limit as $\|\Delta\| \to 0$ $(n \to \infty)$, you obtain

$$\lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\,\Delta x.$$

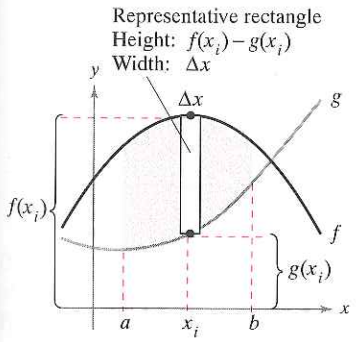


Figure 6.3

Because f and g are continuous on $[a, b]$, $f - g$ is also continuous on $[a, b]$ and the limit exists. Therefore, the area of the given region is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\,\Delta x$$

$$= \int_a^b [f(x) - g(x)]\,dx.$$

Area of a Region Between Two Curves

If f and g are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

In Figure 6.1, the graphs of f and g are shown above the x-axis. This, however is not necessary. The same integrand $[f(x) - g(x)]$ can be used as long as f and g are continuous and $g(x) \leq f(x)$ on the interval $[a, b]$. This result is summarized graphically in Figure 6.4.

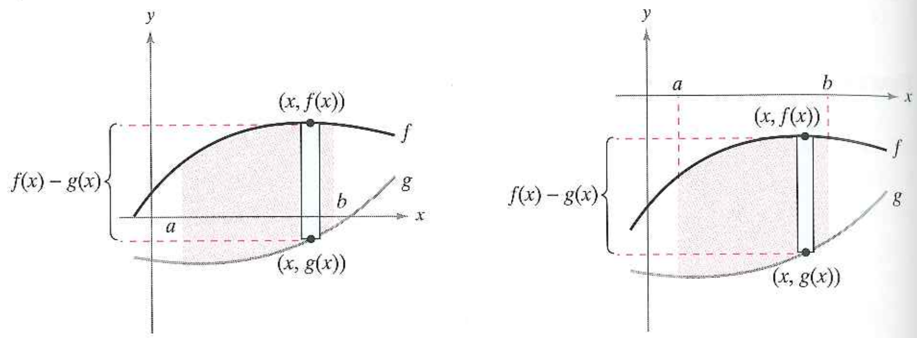


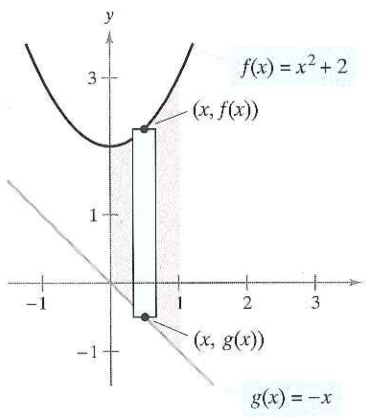
Figure 6.4

NOTE The height of a representative rectangle is $f(x) - g(x)$ regardless of the relative position of the x-axis, as shown in Figure 6.4.

Representative rectangles are used throughout this chapter in various applications of integration. A vertical rectangle (of width Δx) implies integration with respect to x, whereas a horizontal rectangle (of width Δy) implies integration with respect to y.

EXAMPLE 1 Finding the Area of a Region Between Two Curves

Find the area of the region bounded by the graphs of $y = x^2 + 2$, $y = -x$, $x = 0$, and $x = 1$.



Region bounded by the graph of f, the graph of g, $x = 0$, and $x = 1$
Figure 6.5

Solution Let $g(x) = -x$ and $f(x) = x^2 + 2$. Then $g(x) \leq f(x)$ for all x in $[0, 1]$, as shown in Figure 6.5. Thus, the area of the representative rectangle is

$$\Delta A = [f(x) - g(x)]\,\Delta x = [(x^2 + 2) - (-x)]\,\Delta x$$

and the area of the region is

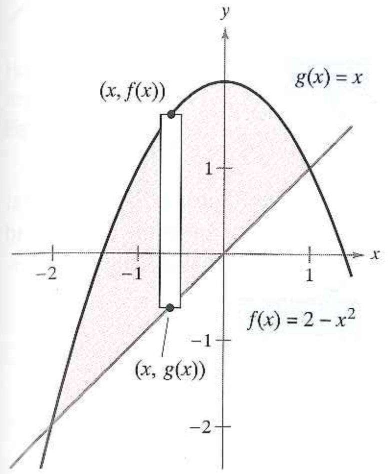
$$\begin{aligned} A = \int_a^b [f(x) - g(x)]\, dx &= \int_0^1 [(x^2 + 2) - (-x)]\, dx \\ &= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_0^1 \\ &= \frac{1}{3} + \frac{1}{2} + 2 \\ &= \frac{17}{6}. \end{aligned}$$

Area of a Region Between Intersecting Curves

In Example 1, the graphs of $f(x) = x^2 + 2$ and $g(x) = -x$ do not intersect, and the values of a and b are given explicitly. A more common problem involves the area of a region bounded by two *intersecting* graphs, where the values of a and b must be calculated.

EXAMPLE 2 A Region Lying Between Two Intersecting Graphs

Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.



Region bounded by the graph of f and the graph of g
Figure 6.6

Solution In Figure 6.6, notice that the graphs of f and g have two points of intersection. To find the x-coordinates of these points, set $f(x)$ and $g(x)$ equal to each other and solve for x.

$$\begin{aligned} 2 - x^2 &= x && \text{Set } f(x) \text{ equal to } g(x). \\ -x^2 - x + 2 &= 0 && \text{Write in standard form.} \\ -(x + 2)(x - 1) &= 0 && \text{Factor.} \\ x &= -2 \text{ or } 1 && \text{Solve for } x. \end{aligned}$$

Thus, $a = -2$ and $b = 1$. Because $g(x) \le f(x)$ on the interval $[-2, 1]$, the representative rectangle has an area of

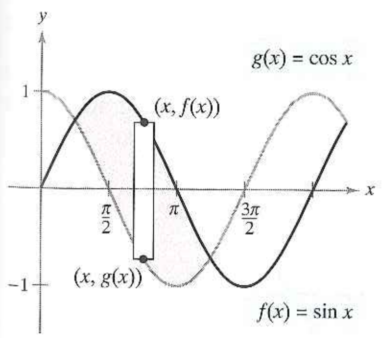
$$\Delta A = [f(x) - g(x)]\,\Delta x = [(2 - x^2) - x]\,\Delta x$$

and the area of the region is

$$A = \int_{-2}^{1} [(2 - x^2) - x]\,dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1}$$
$$= \frac{9}{2}.$$

EXAMPLE 3 A Region Lying Between Two Intersecting Graphs

The sine and cosine curves intersect infinitely many times, bounding regions of equal areas, as shown in Figure 6.7. Find the area of one of these regions.



One of the regions bounded by the graphs of the sine and cosine functions
Figure 6.7

Solution

$$\begin{aligned} \sin x &= \cos x && \text{Set } f(x) \text{ equal to } g(x). \\ \frac{\sin x}{\cos x} &= 1 && \text{Divide both sides by } \cos x. \\ \tan x &= 1 && \text{Trigonometric identity} \\ x &= \frac{\pi}{4} \text{ or } \frac{5\pi}{4}, \quad 0 \le x \le 2\pi && \text{Solve for } x. \end{aligned}$$

Thus, $a = \pi/4$ and $b = 5\pi/4$. Because $\sin x \ge \cos x$ on the interval $[\pi/4, 5\pi/4]$, the area of the region is

$$A = \int_{\pi/4}^{5\pi/4} [\sin x - \cos x]\,dx = \Big[-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4}$$
$$= 2\sqrt{2}.$$

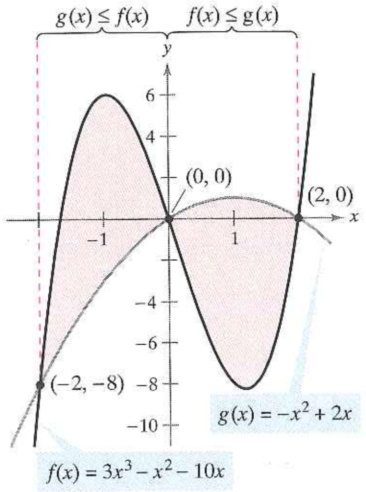
If two curves intersect at *more* than two points, then to find the area of the region between the curves, you must find all points of intersection and check to see which curve is above the other in each interval determined by these points.

EXAMPLE 4 Curves That Intersect at More than Two Points

Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.

Solution Begin by setting $f(x)$ and $g(x)$ equal to each other and solving for x. This yields the x-values of each point of intersection of the two graphs.

$$3x^3 - x^2 - 10x = -x^2 + 2x \qquad \text{Set } f(x) \text{ equal to } g(x).$$
$$3x^3 - 12x = 0 \qquad \text{Write in standard form.}$$
$$3x(x^2 - 4) = 0 \qquad \text{Factor.}$$
$$x = -2, 0, 2 \qquad \text{Solve for } x.$$



In $[-2, 0]$, $g(x) \le f(x)$, and in $[0, 2]$, $f(x) \le g(x)$.
Figure 6.8

Thus, the two graphs intersect when $x = -2$, 0, and 2. In Figure 6.8, notice that $g(x) \le f(x)$ on the interval $[-2, 0]$. However, the two graphs switch at the origin, and $f(x) \le g(x)$ on the interval $[0, 2]$. Hence, you need two integrals—one for the interval $[-2, 0]$ and one for $[0, 2]$.

$$\begin{aligned} A &= \int_{-2}^{0} [f(x) - g(x)]\,dx + \int_{0}^{2} [g(x) - f(x)]\,dx \\ &= \int_{-2}^{0} (3x^3 - 12x)\,dx + \int_{0}^{2} (-3x^3 + 12x)\,dx \\ &= \left[\frac{3x^4}{4} - 6x^2\right]_{-2}^{0} + \left[\frac{-3x^4}{4} + 6x^2\right]_{0}^{2} \\ &= -(12 - 24) + (-12 + 24) \\ &= 24 \end{aligned}$$

NOTE In Example 4, notice that you get an incorrect result if you integrate from -2 to 2. Such integration produces

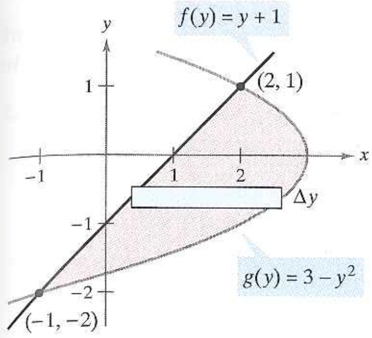
$$\int_{-2}^{2} [f(x) - g(x)]\,dx = \int_{-2}^{2} (3x^3 - 12x)\,dx = 0.$$

If the graph of a function of y is a boundary of a region, it is often convenient to use representative rectangles that are *horizontal* and find the area by integrating with respect to y. In general, to determine the area between two curves, you can use

$$A = \int_{x_1}^{x_2} \underbrace{[(\text{top curve}) - (\text{bottom curve})]}_{\text{in variable } x}\,dx \qquad \text{Vertical rectangles}$$

$$A = \int_{y_1}^{y_2} \underbrace{[(\text{right curve}) - (\text{left curve})]}_{\text{in variable } y}\,dy \qquad \text{Horizontal rectangles}$$

where (x_1, y_1) and (x_2, y_2) are either adjacent points of intersection of the two curves involved or points on the specified boundary lines.



Horizontal rectangles (integration with respect to y)
Figure 6.9

EXAMPLE 5 Horizontal Representative Rectangles

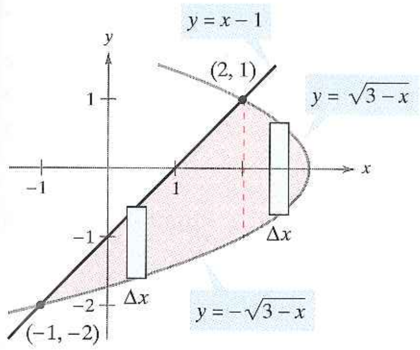
Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.

Solution Consider $g(y) = 3 - y^2$ and $f(y) = y + 1$. These two curves intersect when $y = -2$ and $y = 1$, as shown in Figure 6.9. Because $f(y) \le g(y)$ on this interval, you have

$$\Delta A = [g(y) - f(y)]\,\Delta y = [(3 - y^2) - (y + 1)]\,\Delta y.$$

Hence, the area is

$$\begin{aligned} A &= \int_{-2}^{1} [(3 - y^2) - (y + 1)]\,dy \\ &= \int_{-2}^{1} (-y^2 - y + 2)\,dy \\ &= \left[\frac{-y^3}{3} - \frac{y^2}{2} + 2y\right]_{-2}^{1} \\ &= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) \\ &= \frac{9}{2}. \end{aligned}$$



Vertical rectangles (integration with respect to x)
Figure 6.10

NOTE In Example 5, notice that by integrating with respect to y we need only one integral. If we had integrated with respect to x, we would have needed two integrals

$$A = \int_{-1}^{2} \left[(x - 1) + \sqrt{3 - x}\right] dx + \int_{2}^{3} \left(\sqrt{3 - x} + \sqrt{3 - x}\right) dx$$

because the upper boundary changes at $x = 2$, as shown in Figure 6.10.

In this section, we developed the integration formula for the area between two curves by using a rectangle as the *representative element.* For each new application in the remaining sections of this chapter, we will construct an appropriate representative element using precalculus formulas you already know. Each integration formula then will be obtained by summing or accumulating these representative elements.

Known precalculus formula ⇒ Representative element ⇒ New integration formula

For example, in this section we developed the area formula as follows.

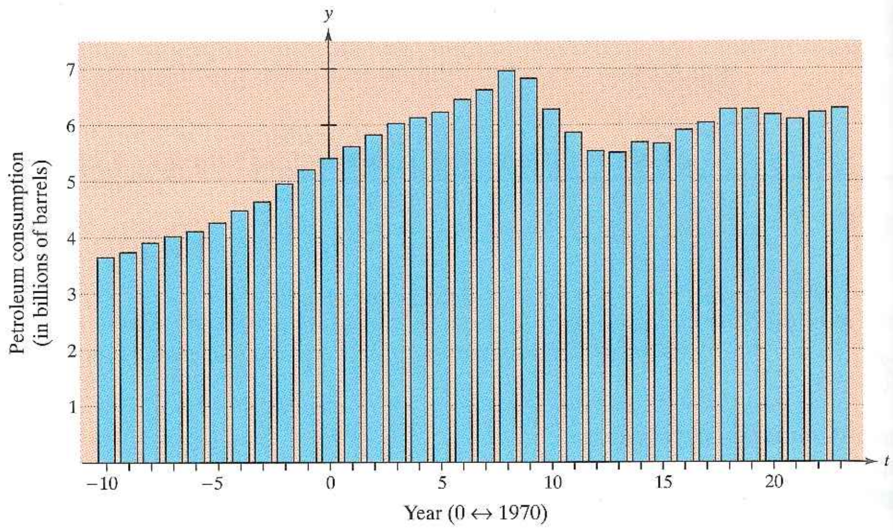
$A = (\text{height})(\text{width})$ Formula for area

↓

$\Delta A = [f(x) - g(x)]\,\Delta x$ Representative area element

↓

$A = \int_a^b [f(x) - g(x)]\,dx$ Accumulate total area by integrating.



From 1960 to 1979, the consumption of petroleum in the United States followed a pattern that was approximately linear. In the late 1970s, however, when crude oil prices increased dramatically, the consumption pattern changed, as shown in the bar graph. *(Source: U.S. Energy Information Administration)*

Figure 6.11

EXAMPLE 6 Consumption of Petroleum

Petroleum consumption patterns have changed in recent years (see Figure 6.11). The rate of consumption (in billions of barrels per year) of petroleum in the United States from 1960 to 1979 can be modeled by

$$f(t) = 0.18t + 5.38, \qquad -10 \le t \le 9$$

where $t = 0$ corresponds to the year starting January 1, 1970. From 1979 through 1993, the rate of consumption can be modeled by

$$g(t) = -0.0029t^3 + 0.149t^2 - 2.42t + 18.38, \qquad 9 \le t \le 23$$

as shown in Figure 6.12. Find the total amount of fuel saved from 1979 through 1993 as a result of fuel being consumed at the post-1979 rate rather than at the pre-1979 rate.

Solution Because the graph of the pre-1979 model lies above the post-1979 graph on the interval $[9, 23]$, the amount of petroleum saved is given by

$$\text{Fuel saved} = \int_9^{23} [f(t) - g(t)]\, dt.$$

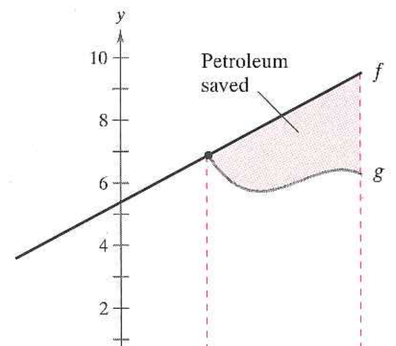
Using the given models for f and g, you can find this amount to be

$$\text{Fuel saved} = \int_9^{23} (0.0029t^3 - 0.149t^2 + 2.6t - 13)\, dt$$

$$= \Big[0.000725t^4 - 0.049667t^3 + 1.3t^2 - 13t\Big]_9^{23}$$

$$\approx 30.44 \text{ billion barrels.}$$

Therefore, approximately 30.44 billion barrels of petroleum were saved.



f: Pre-1979 consumption rate
g: Post-1979 consumption rate
Figure 6.12