SECTION 4.1 Antiderivatives and Indefinite Integration

Antiderivatives • Notation for Antiderivatives • Basic Integration Rules • Initial Conditions and Particular Solutions

Antiderivatives

Suppose you were asked to find a function F whose derivative is

$$f(x) = 3x^2.$$

From your knowledge of derivatives, you would probably say that

$$F(x) = x^3 \text{ because } \frac{d}{dx}[x^3] = 3x^2.$$

Good def

The function F is an *antiderivative* of f. In general, a function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Note that F is called *an* antiderivative of f, rather than *the* antiderivative of f. To see why, observe that

$$F_1(x) = x^3,\ F_2(x) = x^3 - 5, \text{ and } F_3(x) = x^3 + 97$$

are all antiderivatives of $f(x) = 3x^2$. In fact, for any constant C, the function given by $F(x) = x^3 + C$ is an antiderivative of f.

EXPLORATION

Finding Antiderivatives For each of the following derivatives, describe the original function F.

(a) $F'(x) = 2x$

(b) $F'(x) = x$

(c) $F'(x) = x^2$

(d) $F'(x) = \dfrac{1}{x^2}$

(e) $F'(x) = \dfrac{1}{x^3}$

(f) $F'(x) = \cos x$

What strategy did you use to find F?

THEOREM 4.1 Representation of Antiderivatives

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form

$$G(x) = F(x) + C, \text{ for all } x \text{ in } I$$

where C is a constant.

Proof The proof of one direction is straightforward. That is, if $G(x) = F(x) + C$, $F'(x) = f(x)$, and C is a constant, then

$$G'(x) = \frac{d}{dx}[F(x) + C] = F'(x) + 0 = f(x).$$

To prove the other direction, you can define a function H such that

$$H(x) = G(x) - F(x).$$

If H is not constant on the interval I, there must exist a and b $(a < b)$ in the interval such that $H(a) \neq H(b)$. Moreover, because H is differentiable on (a, b), you can apply the Mean Value Theorem to conclude that there exists some c in (a, b) such that

$$H'(c) = \frac{H(b) - H(a)}{b - a}.$$

Because $H(b) \neq H(a)$, it follows that $H'(c) \neq 0$. However, because $G'(c) = F'(c)$, you know that $H'(c) = G'(c) - F'(c) = 0$, which contradicts the fact that $H'(c) \neq 0$. Consequently, you can conclude that $H(x)$ is a constant, C. Therefore, $G(x) - F(x) = C$ and it follows that $G(x) = F(x) + C$.

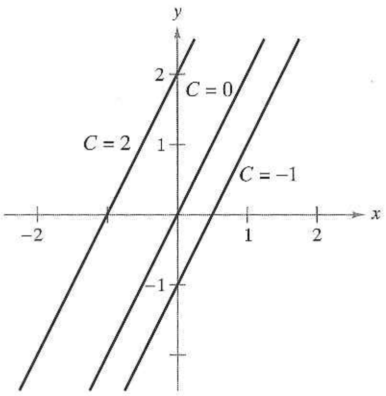
Using Theorem 4.1, you can represent the entire family of antiderivatives of a function by adding a constant to a *known* antiderivative. For example, knowing that $D_x[x^2] = 2x$, you can represent the family of *all* antiderivatives of $f(x) = 2x$ by

$$G(x) = x^2 + C \qquad \text{Family of all antiderivatives of } f(x) = 2x$$

where C is a constant. The constant C is called the **constant of integration**. The family of functions represented by G is the **general antiderivative** of f, and $G(x) = x^2 + C$ is the **general solution** of the *differential equation*

$$G'(x) = 2x. \qquad \text{Differential equation}$$

A **differential equation** in x and y is an equation that involves x, y, and derivatives of y. For instance, $y' = 3x$ and $y' = x^2 + 1$ are examples of differential equations.



Functions of the form $y = 2x + C$
Figure 4.1

EXAMPLE 1 Solving a Differential Equation

Find the general solution of the differential equation $y' = 2$.

Solution To begin, you need to find a function whose derivative is 2. One such function is

$$y = 2x.$$

Now, you can use Theorem 4.1 to conclude that the general solution of the differential equation is

$$y = 2x + C.$$

The graphs of several functions of the form $y = 2x + C$ are shown in Figure 4.1.

Notation for Antiderivatives

When solving a differential equation of the form

$$\frac{dy}{dx} = f(x)$$

it is convenient to write it in the equivalent differential form

$$dy = f(x)\,dx.$$

The operation of finding all solutions of this equation is called **antidifferentiation** (or **indefinite integration**) and is denoted by an integral sign $\int$. The general solution is denoted by

Variable of integration — Constant of integration

$$y = \int f(x)\,dx = F(x) + C.$$

Integrand

The expression $\int f(x)\,dx$ is read as the *antiderivative of f with respect to x*. Thus, the differential dx serves to identify x as the variable of integration. The term **indefinite integral** is a synonym for antiderivative.

In this text, whenever we write $\int f(x)\,dx = F(x) + C$, we mean that F is an antiderivative of f *on an interval*.

Basic Integration Rules

The inverse nature of integration and differentiation can be verified by substituting $F'(x)$ for $f(x)$ in the indefinite integration definition to obtain

$$\int F'(x)\,dx = F(x) + C.$$

Integration is the "inverse" of differentiation.

Moreover, if $\int f(x)\,dx = F(x) + C$, then

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

Differentiation is the "inverse" of integration.

These two equations allow you to obtain integration formulas directly from differentiation formulas, as shown in the following summary.

Basic Integration Rules

Memorize

Differentiation Formula	*Integration Formula*	
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$	
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$	
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$	
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$	
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$	Power Rule
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$	
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\,dx = -\cos x + C$	
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$	
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$	
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$	
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$	

NOTE Note that the Power Rule for integration has the restriction that $n \neq -1$. The evaluation of $\int 1/x\,dx$ must wait until the introduction of the natural logarithm function in Chapter 5.

EXAMPLE 2 Applying the Basic Integration Rules

Describe the antiderivatives of $3x$.

Solution

$$\int 3x\,dx = 3\int x\,dx \qquad \text{Constant Multiple Rule}$$

$$= 3\int x^1\,dx \qquad \text{Rewrite } (x = x^1).$$

$$= 3\left(\frac{x^2}{2}\right) + C \qquad \text{Power Rule } (n = 1)$$

$$= \frac{3}{2}x^2 + C \qquad \text{Simplify.}$$

When indefinite integrals are evaluated, a strict application of the basic integration rules tends to produce complicated constants of integration. For instance, in Example 2, we could have written

$$\int 3x\,dx = 3\int x\,dx$$

$$= 3\left(\frac{x^2}{2} + C\right)$$

$$= \frac{3}{2}x^2 + 3C.$$

However, because C represents *any* constant, it is both cumbersome and unnecessary to write $3C$ as the constant of integration, and we choose the simpler form, $\frac{3}{2}x^2 + C$.

In Example 2, note that the general pattern of integration is similar to that of differentiation.

Original integral ⇨ Rewrite ⇨ Integrate ⇨ Simplify

EXAMPLE 3 Rewriting Before Integrating

	Original Integral	*Rewrite*	*Integrate*	*Simplify*
a.	$\int \frac{1}{x^3}\,dx$	$\int x^{-3}\,dx$	$\frac{x^{-2}}{-2} + C$	$-\frac{1}{2x^2} + C$
b.	$\int \sqrt{x}\,dx$	$\int x^{1/2}\,dx$	$\frac{x^{3/2}}{3/2} + C$	$\frac{2}{3}x^{3/2} + C$
c.	$\int 2\sin x\,dx$	$2\int \sin x\,dx$	$2(-\cos x) + C$	$-2\cos x + C$

TECHNOLOGY Some software programs such as *Derive, Maple, Mathcad, Mathematica*, and the *TI-92* are capable of performing integration symbolically. If you have access to such a symbolic integration utility, try using it to evaluate the indefinite integrals in Example 3.

Remember that you can check your answer to an antidifferentiation problem by differentiating. For instance, in Example 3b, you can check that $\frac{2}{3}x^{3/2} + C$ is the correct antiderivative by differentiating the answer to obtain

$$D_x\left[\frac{2}{3}x^{3/2} + C\right] = \left(\frac{2}{3}\right)\left(\frac{3}{2}\right)x^{1/2} = \sqrt{x}.$$

The basic integration rules listed earlier in this section allow you to integrate *any* polynomial function, as demonstrated in Example 4.

EXAMPLE 4 Integrating Polynomial Functions

a. $\int dx = \int 1\,dx$

$= x + C$

b. $\int (x + 2)\,dx = \int x\,dx + \int 2\,dx$

$= \frac{x^2}{2} + C_1 + 2x + C_2$

$= \frac{x^2}{2} + 2x + C$ $\qquad C = C_1 + C_2$

The second line in the solution is usually omitted.

c. $\int (3x^4 - 5x^2 + x)\,dx = 3\left(\frac{x^5}{5}\right) - 5\left(\frac{x^3}{3}\right) + \frac{x^2}{2} + C$

$= \frac{3}{5}x^5 - \frac{5}{3}x^3 + \frac{1}{2}x^2 + C$

EXAMPLE 5 Rewriting Before Integrating

$\int \frac{x+1}{\sqrt{x}}\,dx = \int \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx$ Rewrite as two fractions.

$= \int (x^{1/2} + x^{-1/2})\,dx$ Rewrite with fractional exponents.

$= \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C$ Integrate.

$= \frac{2}{3}x^{3/2} + 2x^{1/2} + C$ Simplify.

NOTE When integrating quotients, do not integrate the numerator and denominator separately. This is no more valid in integration than it is in differentiation. For instance, in Example 5, be sure you understand that

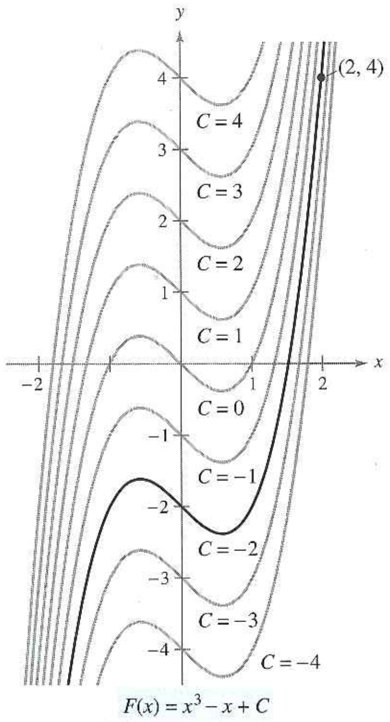
$$\int \frac{x+1}{\sqrt{x}}\,dx \neq \frac{\int (x+1)\,dx}{\int \sqrt{x}\,dx}.$$

EXAMPLE 6 Rewriting Before Integrating

$\int \frac{\sin x}{\cos^2 x}\,dx = \int \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) dx$ Rewrite as a product.

$= \int \sec x \tan x\,dx$ Rewrite using trigonometric identities.

$= \sec x + C$ Integrate.



The particular solution that satisfies the initial condition $F(2) = 4$ is $F(x) = x^3 - x - 2$.
Figure 4.2

Initial Conditions and Particular Solutions

You have already seen that the equation $y = \int f(x)\,dx$ has many solutions (each differing from the others by a constant). This means that the graphs of any two antiderivatives of f are vertical translations of each other. For example, Figure 4.2 shows the graphs of several antiderivatives of the form

$$y = \int (3x^2 - 1)\,dx = x^3 - x + C \qquad \text{General solution}$$

for various integer values of C. Each of these antiderivatives is a solution of the differential equation

$$\frac{dy}{dx} = 3x^2 - 1.$$

In many applications of integration, you are given enough information to determine a **particular solution**. To do this you need only know the value of $y = F(x)$ for one value of x. (This information is called an **initial condition**.) For example, in Figure 4.2, only one curve passes through the point (2, 4). To find this curve, you can use the following information.

$$F(x) = x^3 - x + C \qquad \text{General solution}$$
$$F(2) = 4 \qquad \text{Initial condition}$$

By using the initial condition in the general solution, you can determine that $F(2) = 8 - 2 + C = 4$, which implies that $C = -2$. Thus, you obtain

$$F(x) = x^3 - x - 2. \qquad \text{Particular solution}$$

EXAMPLE 7 Finding a Particular Solution

Find the general solution of

$$F'(x) = \frac{1}{x^2}, \quad x > 0$$

and find the particular solution that satisfies the initial condition $F(1) = 0$.

Solution To find the general solution, integrate to obtain

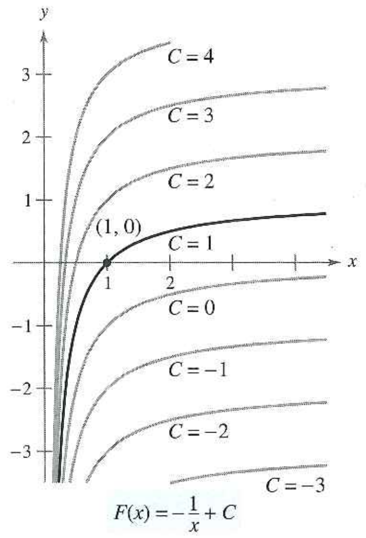
$$F(x) = \int \frac{1}{x^2}\,dx$$
$$= \int x^{-2}\,dx$$
$$= \frac{x^{-1}}{-1} + C$$
$$= -\frac{1}{x} + C. \qquad \text{General solution}$$

Using the initial condition $F(1) = 0$, you can solve for C as follows.

$$F(1) = -\frac{1}{1} + C = 0 \implies C = 1$$

Thus, the particular solution, as shown in Figure 4.3, is

$$F(x) = -\frac{1}{x} + 1, \quad x > 0. \qquad \text{Particular solution}$$

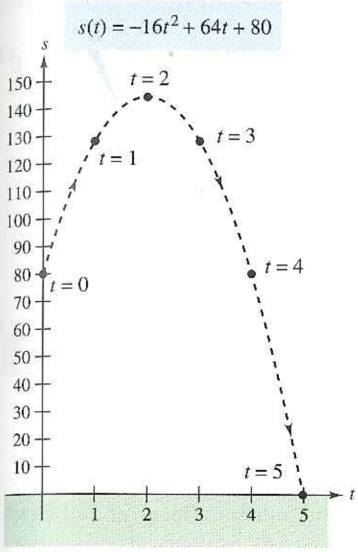


The particular solution that satisfies the initial condition $F(1) = 0$ is $F(x) = -(1/x) + 1, x > 0$.
Figure 4.3

So far in this section we have been using x as the variable of integration. In applications, it is often convenient to use a different variable. For instance, in the following example involving *time*, the variable of integration is t.

EXAMPLE 8 Solving a Vertical Motion Problem

A ball is thrown upward with an initial velocity of 64 feet per second from an initial height of 80 feet, as shown in Figure 4.4.



Position of a ball at time t
Figure 4.4

NOTE In Example 8, note that the position function has the form

$$s(t) = \tfrac{1}{2}gt^2 + v_0t + s_0,$$

where $g = -32$, v_0 is the initial velocity, and s_0 is the initial height, as presented in Section 2.2.

a. Find the position function giving the height s as a function of the time t.

b. When does the ball hit the ground?

Solution

a. Let $t = 0$ represent the initial time. The two given initial conditions can be written as follows.

$$s(0) = 80 \qquad \text{Initial height is 80 feet.}$$
$$s'(0) = 64 \qquad \text{Initial velocity is 64 feet per second.}$$

Using -32 feet per second per second as the acceleration due to gravity, you can write

$$s''(t) = -32$$
$$s'(t) = \int s''(t)\,dt = \int -32\,dt = -32t + C_1.$$

Using the initial velocity, you obtain $s'(0) = 64 = -32(0) + C_1$, which implies that $C_1 = 64$. Next, by integrating $s'(t)$, you obtain

$$s(t) = \int s'(t)\,dt = \int (-32t + 64)\,dt = -16t^2 + 64t + C_2$$

and using the initial height, you obtain

$$s(0) = 80 = -16(0^2) + 64(0) + C_2$$

which implies that $C_2 = 80$. Therefore, the position function is

$$s(t) = -16t^2 + 64t + 80.$$

b. Using the position function found in part (a), you can find the time that the ball hits the ground by solving the equation $s(t) = 0$.

$$s(t) = -16t^2 + 64t + 80 = 0$$
$$-16(t + 1)(t - 5) = 0$$
$$t = -1, 5$$

Because t must be positive, you can conclude that the ball hits the ground 5 seconds after it was thrown.

Example 8 shows how to use calculus to analyze vertical motion problems in which the acceleration is determined by a gravitational force. You can use a similar strategy to analyze other linear motion problems (vertical or horizontal) in which the acceleration (or deceleration) is the result of some other force, as you can see in Exercises 67–76.

Before you begin the exercise set, be sure you realize that one of the most important steps in integration is *rewriting the integrand* in a form that fits the basic integration rules. To further illustrate this point, here are some additional examples.

Original Integral	*Rewrite*	*Integrate*	*Simplify*
$\int \frac{2}{\sqrt{x}}\,dx$	$2\int x^{-1/2}\,dx$	$2\left(\frac{x^{1/2}}{1/2}\right) + C$	$4x^{1/2} + C$
$\int (t^2+1)^2\,dt$	$\int (t^4 + 2t^2 + 1)\,dt$	$\frac{t^5}{5} + 2\left(\frac{t^3}{3}\right) + t + C$	$\frac{1}{5}t^5 + \frac{2}{3}t^3 + t + C$
$\int \frac{x^3+3}{x^2}\,dx$	$\int (x + 3x^{-2})\,dx$	$\frac{x^2}{2} + 3\left(\frac{x^{-1}}{-1}\right) + C$	$\frac{1}{2}x^2 - \frac{3}{x} + C$
$\int \sqrt[3]{x}(x-4)\,dx$	$\int (x^{4/3} - 4x^{1/3})\,dx$	$\frac{x^{7/3}}{7/3} - 4\left(\frac{x^{4/3}}{4/3}\right) + C$	$\frac{3}{7}x^{4/3}(x-7) + C$

EXERCISES FOR SECTION 4.1

In Exercises 1–4, verify the statement by showing that the derivative of the right side is equal to the integrand of the left side.

1. $\int \left(-\frac{9}{x^4}\right) dx = \frac{3}{x^3} + C$
2. $\int \left(4x^3 - \frac{1}{x^2}\right) dx = x^4 + \frac{1}{x} + C$
3. $\int (x-2)(x+2)\,dx = \frac{1}{3}x^3 - 4x + C$
4. $\int \frac{x^2-1}{x^{3/2}}\,dx = \frac{2(x^2+3)}{3\sqrt{x}} + C$

In Exercises 5–10, complete the table using the examples at the top of this page as a model.

Original Integral	*Rewrite*	*Integrate*	*Simplify*
5. $\int \sqrt[3]{x}\,dx$			
6. $\int \frac{1}{x^2}\,dx$			
7. $\int \frac{1}{x\sqrt{x}}\,dx$			
8. $\int x(x^2+3)\,dx$			
9. $\int \frac{1}{2x^3}\,dx$			
10. $\int \frac{1}{(2x)^3}\,dx$			

In Exercises 11–14, find the general solution of the differential equation and check the result by differentiation.

11. $\frac{dy}{dt} = 3t^2$
12. $\frac{dr}{d\theta} = \pi$
13. $\frac{dy}{dx} = x^{3/2}$
14. $\frac{dy}{dx} = 3x^{-4}$

In Exercises 15–30, evaluate the indefinite integral and check the result by differentiation.

15. $\int (x^3 + 2)\,dx$
16. $\int (x^2 - 2x + 3)\,dx$
17. $\int (x^{3/2} + 2x + 1)\,dx$
18. $\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx$
19. $\int \sqrt[3]{x^2}\,dx$
20. $\int (\sqrt[4]{x^3} + 1)\,dx$
21. $\int \frac{1}{x^3}\,dx$
22. $\int \frac{1}{x^4}\,dx$
23. $\int \frac{x^2+x+1}{\sqrt{x}}\,dx$
24. $\int \frac{x^2+1}{x^2}\,dx$
25. $\int (x+1)(3x-2)\,dx$
26. $\int (2t^2-1)^2\,dt$
27. $\int y^2\sqrt{y}\,dy$
28. $\int (1+3t)t^2\,dt$
29. $\int dx$
30. $\int 3\,dt$

In Exercises 31–38, evaluate the trigonometric integral and check the result by differentiation.

31. $\int (2\sin x + 3\cos x)\,dx$
32. $\int (t^2 - \sin t)\,dt$
33. $\int (1 - \csc t \cot t)\,dt$
34. $\int (\theta^2 + \sec^2\theta)\,d\theta$
35. $\int (\sec^2\theta - \sin\theta)\,d\theta$
36. $\int \sec y(\tan y - \sec y)\,dy$
37. $\int (\tan^2 y + 1)\,dy$
38. $\int \frac{\sin x}{1 - \sin^2 x}\,dx$

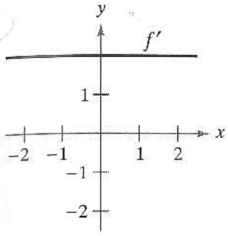
In Exercises 39 and 40, sketch the graphs of the function $g(x) = f(x) + C$ for $C = -2$, $C = 0$, and $C = 3$ on the same set of coordinate axes.

39. $f(x) = \cos x$

40. $f(x) = \sqrt{x}$

In Exercises 41–44, the graph of the derivative of a function is given. Sketch the graphs of *two* functions that have the given derivative. (There is more than one correct answer.)

41.



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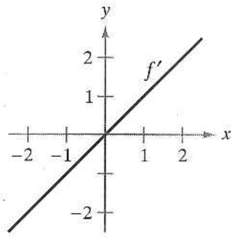

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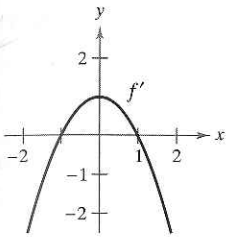

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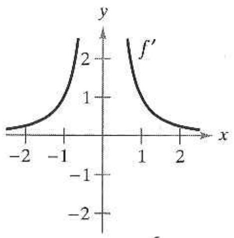
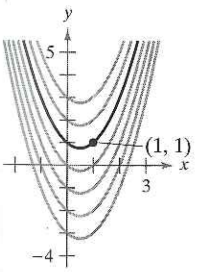
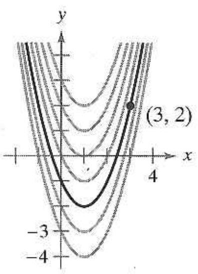


In Exercises 45–48, find the equation for y, given the derivative and the indicated point on the curve.

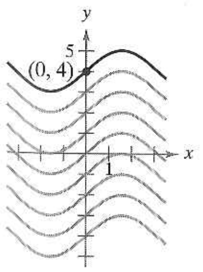
45. $\dfrac{dy}{dx} = 2x - 1$



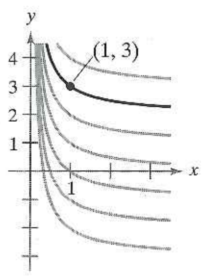
46. $\dfrac{dy}{dx} = 2(x - 1)$



47. $\dfrac{dy}{dx} = \cos x$

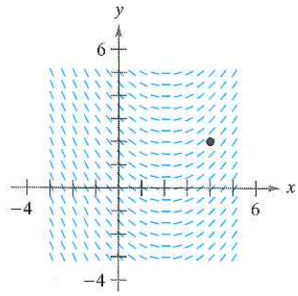


48. $\dfrac{dy}{dx} = -\dfrac{1}{x^2}, \quad x > 0$

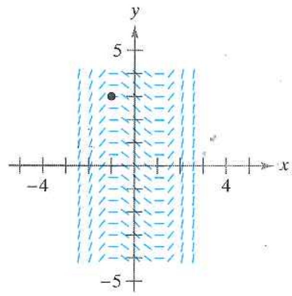


Direction Fields **In Exercises 49 and 50, a differential equation, a point, and a direction field are given. A *direction field* consists of line segments with slopes given by the differential equation. These line segments give a visual perspective of the directions of the solutions of the differential equation. (a) Sketch two approximate solutions of the differential equation on the direction field, one of which passes through the indicated point. (b) Use integration to find the particular solution of the differ- ential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a).**

49. $\dfrac{dy}{dx} = \dfrac{1}{2}x - 1, \; (4, 2)$



50. $\dfrac{dy}{dx} = x^2 - 1, \; (-1, 3)$



In Exercises 51–54, solve the differential equation.

51. $f''(x) = 2, \; f'(2) = 5, \; f(2) = 10$

52. $f''(x) = x^2, \; f'(0) = 6, \; f(0) = 3$

53. $f''(x) = x^{-3/2}, \; f'(4) = 2, \; f(0) = 0$

54. $f''(x) = \sin x, \; f'(0) = 1, \; f(0) = 6$

55. ***Tree Growth*** An evergreen nursery usually sells a certain shrub after 6 years of growth and shaping. The growth rate during those 6 years is approximated by

$$\frac{dh}{dt} = 1.5t + 5$$

where t is the time in years and h is the height in centimeters. The seedlings are 12 centimeters tall when planted ($t = 0$).

(a) Find the height after t years.

(b) How tall are the shrubs when they are sold?

56. ***Population Growth*** The rate of growth dP/dt of a population of bacteria is proportional to the square root of t, where P is the population size and t is the time in days ($0 \le t \le 10$). That is,

$$\frac{dP}{dt} = k\sqrt{t}.$$

The initial size of the population is 500. After 1 day, the population has grown to 600. Estimate the population after 7 days.

Vertical Motion **In Exercises 57–60, use $a(t) = -32$ feet per second per second as the acceleration due to gravity. (Neglect air resistance.)**

57. A ball is thrown vertically upward from the ground with an initial velocity of 60 feet per second. How high will the ball go?

58. Show that the height above the ground of an object thrown upward from a point s_0 feet above the ground with an initial velocity of v_0 feet per second is given by the function

$$f(t) = -16t^2 + v_0 t + s_0.$$

59. With what initial velocity must an object be thrown upward (from ground level) to reach the top of the Washington Monument (approximately 550 feet)?

60. A balloon, rising vertically with a velocity of 16 feet per second, releases a sandbag at the instant it is 64 feet above the ground.

(a) How many seconds after its release will the bag strike the ground?

(b) At what velocity will it hit the ground?

Vertical Motion **In Exercises 61–64, use $a(t) = -9.8$ meters per second per second as the acceleration due to gravity. (Neglect air resistance.)**

61. Show that the height above the ground of an object thrown upward from a point s_0 meters above the ground with an initial velocity of v_0 meters per second is given by the function

$$f(t) = -4.9t^2 + v_0 t + s_0.$$

62. ***Grand Canyon*** The Grand Canyon is 1600 meters deep at its deepest point. A rock is dropped from the rim above this point. Express the height of the rock as a function of the time t in seconds. How long will it take the rock to hit the canyon floor?

63. A baseball is thrown upward from ground level with a velocity of 10 meters per second. Determine its maximum height.

64. With what initial velocity must an object be thrown upward (from ground level) to reach a maximum height of 200 meters?

65. ***Lunar Gravity*** On the moon, the acceleration due to gravity is -1.6 meters per second per second. A stone is dropped from a cliff on the moon and hits the surface of the moon 20 seconds later. How far did it fall? What was its velocity at impact?

66. ***Escape Velocity*** The minimum velocity required for an object to escape earth's gravitational pull is obtained from the solution of the equation

$$\int v\,dv = -GM\int \frac{1}{y^2}\,dy$$

where v is the velocity of the object projected from the earth, y is the distance from the center of the earth, G is the gravitational constant, and M is the mass of the earth.

Show that v and y are related by the equation

$$v^2 = v_0^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

where v_0 is the initial velocity of the object and R is the radius of the earth.

Rectilinear Motion **In Exercises 67–70, consider a particle moving along the x-axis where $x(t)$ is the position of the particle at time t, $x'(t)$ is its velocity, and $x''(t)$ is its acceleration.**

67. $x(t) = t^3 - 6t^2 + 9t - 2, \quad 0 \le t \le 5$

(a) Find the velocity and acceleration of the particle.

(b) Find the open t-intervals on which the particle is moving to the right.

(c) Find the velocity of the particle when the acceleration is 0.

68. Repeat Exercise 67 for the position function

$$x(t) = (t - 1)(t - 3)^2, \quad 0 \le t \le 5.$$

69. A particle moves along the x-axis at a velocity of $v(t) = 1/\sqrt{t}$, $t > 0$. At time $t = 1$, its position is $x = 4$. Find the acceleration and position functions for the particle.

70. A particle, initially at rest, moves along the x-axis such that its acceleration at time $t > 0$ is given by $a(t) = \cos t$. At the time $t = 0$, its position is $x = 3$.

(a) Find the velocity and position functions for the particle.

(b) Find the values of t for which the particle is at rest.

71. ***Acceleration*** The maker of a certain automobile advertises that it takes 13 seconds to accelerate from 25 kilometers per hour to 80 kilometers per hour. Assuming constant acceleration, compute the following.

(a) The acceleration in meters per second per second

(b) The distance the car travels during the 13 seconds

72. ***Deceleration*** A car traveling at 45 miles per hour is brought to a stop, at constant deceleration, 132 feet from where the brakes are applied.

(a) How far has the car moved when its speed has been reduced to 30 miles per hour?

(b) How far has the car moved when its speed has been reduced to 15 miles per hour?

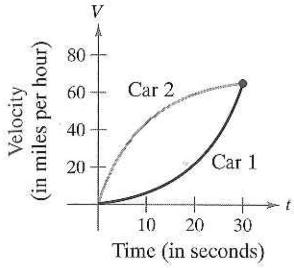
(c) Draw the real number line from 0 to 132, and plot the points found in parts (a) and (b). What can you conclude?

73. ***Acceleration*** At the instant the traffic light turns green, a car that has been waiting at an intersection starts with a constant acceleration of 6 feet per second per second. At the same instant, a truck traveling with a constant velocity of 30 feet per second passes the car.

(a) How far beyond its starting point will the car pass the truck?

(b) How fast will the car be traveling when it passes the truck?

74. *Think About It* Two cars starting from rest accelerate to 65 miles per hour in 30 seconds. The velocity of each car is shown in the figure. Are the cars side by side at the end of the 30-second time interval? Explain.



75. *Acceleration* Assume that a fully loaded plane starting from rest has a constant acceleration while moving down a runway. The plane requires 0.7 mile of runway and a speed of 160 miles per hour in order to lift off. What is the plane's acceleration?

76. *Airplane Separation* Two airplanes are in a straight-line landing pattern and, according to FAA regulations, must keep at least a 3-mile separation. Airplane A is 10 miles from touchdown and is gradually slowing its speed from 150 miles per hour to a landing speed of 100 miles per hour. Airplane B is 17 miles from touchdown and is gradually slowing its speed from 250 miles per hour to a landing speed of 115 miles per hour.

(a) Assuming the deceleration of each airplane is constant, find the position functions s_1 and s_2 for airplane A and airplane B. Let $t = 0$ represent the times when the airplanes are 10 and 17 miles from the airport.

(b) Use a graphing utility to graph the position functions.

(c) Find a formula for the magnitude of the distance d between the two airplanes as a function of t. Use a graphing utility to graph d. Is $d < 3$ for some time prior to the landing of airplane A? If so, find that time.

(d) If the airplanes do not keep the required separation, determine how much the 250 miles per hour speed for airplane B should be reduced in order to meet FAA requirements.

Marginal Cost **In Exercises 77 and 78, find the cost function and average cost for the given marginal cost and fixed cost (cost when $x = 0$).**

	Marginal Cost	*Fixed Cost*
77.	$\frac{dC}{dx} = 2x - 12$	\$50
78.	$\frac{dC}{dx} = \frac{\sqrt[4]{x}}{10} + 10$	\$2300

In Exercises 79 and 80, find the revenue and demand functions for the given marginal revenue.

79. $\frac{dR}{dx} = 100 - 5x$

80. $\frac{dR}{dx} = 100 - 6x - 2x^2$

True or False? **In Exercises 81–84, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

81. Each antiderivative of an nth-degree polynomial function is an $(n + 1)$st-degree polynomial function.

82. If $p(x)$ is a polynomial function, then p has exactly one antiderivative whose graph contains the origin.

83. If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$, then $F(x) = G(x) + C$.

84. If $f'(x) = g(x)$, then $\int g(x)\,dx = f(x) + C$.

85. *Think About It* Use the graph of f' in the figure to answer the following, given that $f(0) = -4$.

(a) Approximate the slope of f at $x = 4$. Explain.

(b) Is it possible that $f(2) = -1$? Explain.

(c) Is $f(5) - f(4) > 0$? Explain.

(d) Approximate the value of x where f is maximum. Explain.

(e) Approximate any intervals in which the graph of f is concave upward and any intervals in which it is concave downward. Approximate the x-coordinates of any points of inflection.

(f) Approximate the x-coordinate of the minimum of $f''(x)$.

(g) Sketch an approximate graph of f.

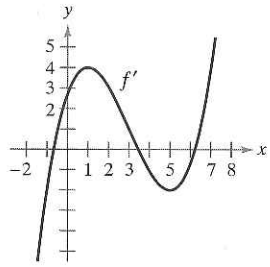


Figure for 85

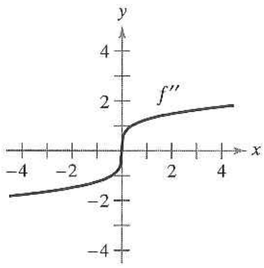


Figure for 86

86. *Think About It* The graphs of f and f' each pass through the origin. Use the graph of f'' shown in the figure to sketch the graphs of f and f'.

87. *Acceleration* Galileo Galilei (1564–1642) stated the following proposition concerning falling objects: The time in which any space is traversed by a uniformly accelerating body is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest speed of the accelerating body and the speed just before acceleration began. Use the techniques of this section to verify this proposition.

88. Let $s(x)$ and $c(x)$ be two functions satisfying $s'(x) = c(x)$ and $c'(x) = -s(x)$ for all x. If $s(0) = 0$ and $c(0) = 1$, prove that $[s(x)]^2 + [c(x)]^2 = 1$.