SECTION 3.6 A Summary of Curve Sketching

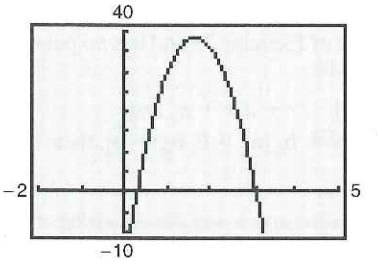
Summary of Curve-Sketching Techniques

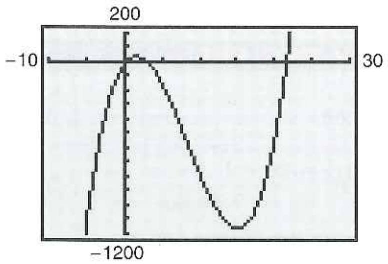
Summary of Curve-Sketching Techniques

It would be difficult to overstate the importance of using graphs in mathematics. Descartes's introduction of analytic geometry contributed significantly to the rapid advances in calculus that began during the mid-seventeenth century. In the words of Lagrange, "As long as algebra and geometry traveled separate paths their advance was slow and their applications limited. But when these two sciences joined company, they drew from each other fresh vitality and thenceforth marched on at a rapid pace toward perfection."

So far, you have studied several concepts that are useful in analyzing the graph of a function.

- x-intercepts and y-intercepts (Section P.1)
- Symmetry (Section P.1)
- Domain and range (Section P.3)
- Continuity (Section 1.4)
- Vertical asymptotes (Section 1.5)
- Differentiability (Section 2.1)
- Relative extrema (Section 3.1)
- Concavity (Section 3.4)
- Points of inflection (Section 3.4)
- Horizontal asymptotes (Section 3.5)





Different viewing rectangles for the same graph
Figure 3.42

When you are sketching the graph of a function, either by hand or with a graphing utility, remember that normally you cannot show the *entire* graph. The decision as to which part of the graph you choose to show is often crucial. For instance, which of the viewing rectangles in Figure 3.42 better represents the graph of

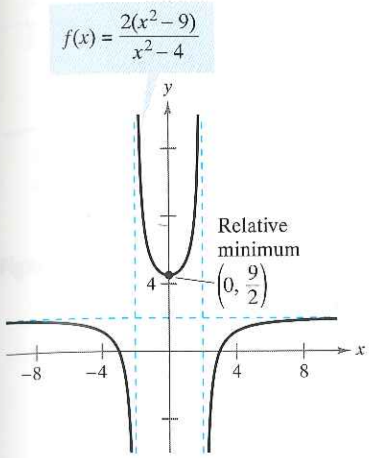
$$f(x) = x^3 - 25x^2 + 74x - 20?$$

By seeing both views, it is clear that the second viewing rectangle gives a more complete representation of the graph. But would a third viewing rectangle reveal other interesting portions of the graph? To answer this, you need to use calculus to interpret the first and second derivatives. Here are some guidelines for determining a good viewing rectangle for the graph of a function.

Guidelines for Analyzing the Graph of a Function

1. Determine the domain and range of the function.
2. Determine the intercepts and asymptotes of the graph.
3. Locate the x-values for which $f'(x)$ and $f''(x)$ are either zero or undefined. Use the results to determine relative extrema and points of inflection.

NOTE In these guidelines, note the importance of *algebra* (as well as calculus) for solving the equations $f(x) = 0$, $f'(x) = 0$, and $f''(x) = 0$.



Using calculus, you can be certain that you have determined all characteristics of the graph of f.
Figure 3.43

FOR FURTHER INFORMATION For more information on the use of technology to graph rational functions, see the article "Graphs of Rational Functions for Computer Assisted Calculus" by Stan Byrd and Terry Walters in the September 1991 issue of *The College Mathematics Journal.*

EXAMPLE 1 Analyzing the Graph of a Rational Function

Analyze the graph of $f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$.

Solution

First derivative: $f'(x) = \dfrac{20x}{(x^2 - 4)^2}$

Second derivative: $f''(x) = \dfrac{-20(3x^2 + 4)}{(x^2 - 4)^3}$

x-intercepts: $(-3, 0), (3, 0)$

y-intercept: $\left(0, \frac{9}{2}\right)$

Vertical asymptotes: $x = -2, x = 2$

Horizontal asymptote: $y = 2$

Critical number: $x = 0$

Possible points of inflection: None

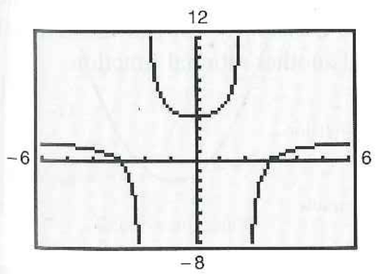
Domain: All real numbers except $x = \pm 2$

Symmetry: With respect to y-axis

Test intervals: $(-\infty, -2), (-2, 0), (0, 2), (2, \infty)$

The table shows how the test intervals are used to determine several characteristics of the graph. The graph of f is shown in Figure 3.43.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < -2$		$-$	$-$	Decreasing, concave down
$x = -2$	Undefined	Undefined	Undefined	Vertical asymptote
$-2 < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	$\frac{9}{2}$	0	$+$	Relative minimum
$0 < x < 2$		$+$	$+$	Increasing, concave up
$x = 2$	Undefined	Undefined	Undefined	Vertical asymptote
$2 < x < \infty$		$+$	$-$	Increasing, concave down



By not using calculus you may overlook important characteristics of the graph of g.
Figure 3.44

Be sure you understand all of the implications of creating a table such as that shown in Example 1. Because of the use of calculus, you can *be sure* that the graph has no relative extrema or points of inflection other than those indicated in Figure 3.43.

Without using the type of analysis outlined in Example 1, it is easy to obtain an incomplete view of a graph's basic characteristics. For instance, Figure 3.44 shows a view of the graph of

$$g(x) = \frac{2(x^2 - 9)(x - 20)}{(x^2 - 4)(x - 21)}.$$

From this view, it appears that the graph of g is about the same as the graph of f shown in Figure 3.43. The graphs of these two functions, however, differ significantly. Try enlarging the viewing rectangle to see the differences.

EXAMPLE 2 Analyzing the Graph of a Rational Function

Analyze the graph of $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$.

Solution

First derivative: $f'(x) = \dfrac{x(x-4)}{(x-2)^2}$

Second derivative: $f''(x) = \dfrac{8}{(x-2)^3}$

x-intercepts: None

y-intercept: $(0, -2)$

Vertical asymptote: $x = 2$

Horizontal asymptotes: None

Critical numbers: $x = 0, x = 4$

Possible points of inflection: None

Domain: All real numbers except $x = 2$

Test intervals: $(-\infty, 0), (0, 2), (2, 4), (4, \infty)$

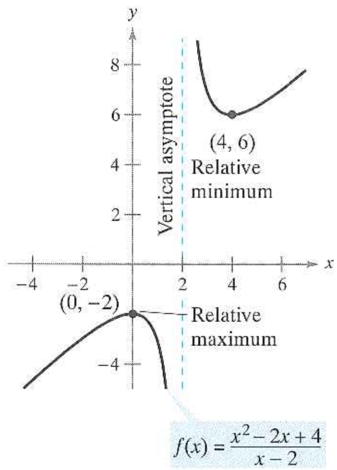


Figure 3.45

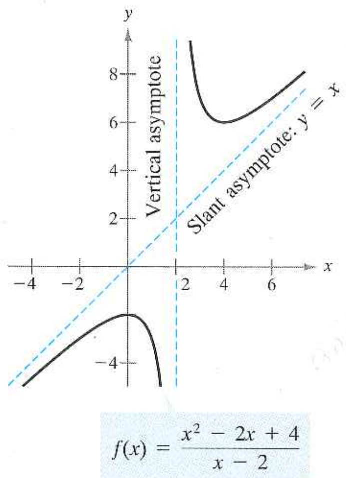
The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.45.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		+	−	Increasing, concave down
$x = 0$	−2	0	−	Relative maximum
$0 < x < 2$		−	−	Decreasing, concave down
$x = 2$	Undefined	Undefined	Undefined	Vertical asymptote
$2 < x < 4$		−	+	Decreasing, concave up
$x = 4$	6	0	+	Relative minimum
$4 < x < \infty$		+	+	Increasing, concave up

Although the graph of the function in Example 2 has no horizontal asymptote, it does have a slant asymptote. The graph of a rational function (having no common factors) has a **slant asymptote** if the degree of the numerator exceeds the degree of the denominator by 1. To find the slant asymptote, use division to rewrite the rational function as the sum of a first-degree polynomial and another rational function.

$$f(x) = \frac{x^2 - 2x + 4}{x - 2} \qquad \text{Rewrite using long division.}$$

$$= x + \frac{4}{x - 2} \qquad y = x \text{ is a slant asymptote.}$$



A slant asymptote
Figure 3.46

In Figure 3.46, note that the graph of f approaches the slant asymptote $y = x$ as x approaches $-\infty$ or ∞. Try using a graphing utility to graph $f(x)$ and the line $y = x$ in the same viewing rectangle.

EXAMPLE 3 Analyzing the Graph of a Radical Function

Analyze the graph of $f(x) = \dfrac{x}{\sqrt{x^2 + 2}}$.

Solution

$$f'(x) = \frac{2}{(x^2 + 2)^{3/2}} \qquad f''(x) = -\frac{6x}{(x^2 + 2)^{5/2}}$$

The graph has only one intercept, $(0, 0)$. It has no vertical asymptotes, but it has two horizontal asymptotes: $y = 1$ (to the right) and $y = -1$ (to the left). The function has no critical numbers and one possible point of inflection (at $x = 0$). The domain of the function is all real numbers, and the graph is symmetric with respect to the origin. The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.47.

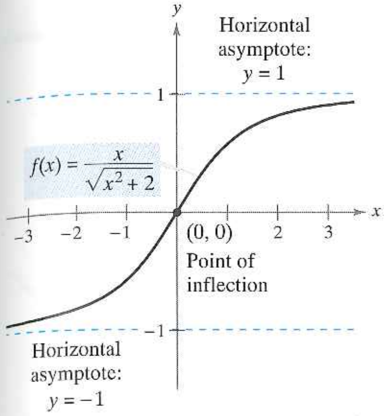


Figure 3.47

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		$+$	$+$	Increasing, concave up
$x = 0$	0	$\frac{1}{\sqrt{2}}$	0	Point of inflection
$0 < x < \infty$		$+$	$-$	Increasing, concave down

EXAMPLE 4 Analyzing the Graph of a Radical Function

Analyze the graph of $f(x) = 2x^{5/3} - 5x^{4/3}$.

Solution

$$f'(x) = \frac{10}{3}x^{1/3}(x^{1/3} - 2) \qquad f''(x) = \frac{20(x^{1/3} - 1)}{9x^{2/3}}$$

The function has two intercepts: $(0, 0)$ and $\left(\frac{125}{8}, 0\right)$. There are no horizontal or vertical asymptotes. The function has two critical numbers ($x = 0$ and $x = 8$) and two possible points of inflection ($x = 0$ and $x = 1$). The domain is all real numbers. The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.48.

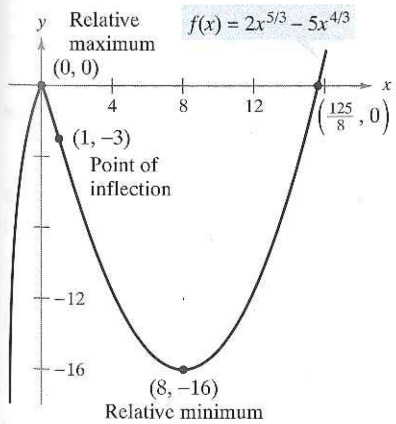


Figure 3.48

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		$+$	$-$	Increasing, concave down
$x = 0$	0	0	Undefined	Relative maximum
$0 < x < 1$		$-$	$-$	Decreasing, concave down
$x = 1$	-3	$-$	0	Point of inflection
$1 < x < 8$		$-$	$+$	Decreasing, concave up
$x = 8$	-16	0	$+$	Relative minimum
$8 < x < \infty$		$+$	$+$	Increasing, concave up

EXAMPLE 5 Analyzing the Graph of a Polynomial Function

Analyze the graph of $f(x) = x^4 - 12x^3 + 48x^2 - 64x$.

Solution Begin by factoring to obtain

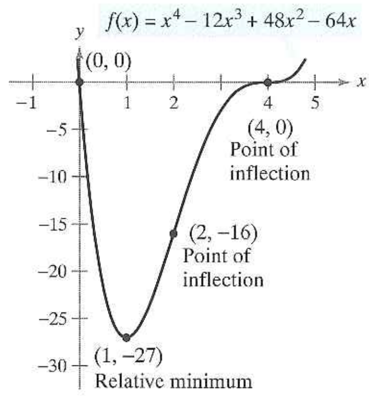
$$f(x) = x^4 - 12x^3 + 48x^2 - 64x$$
$$= x(x - 4)^3.$$

Then, using the factored form of $f(x)$, you can perform the following analysis.

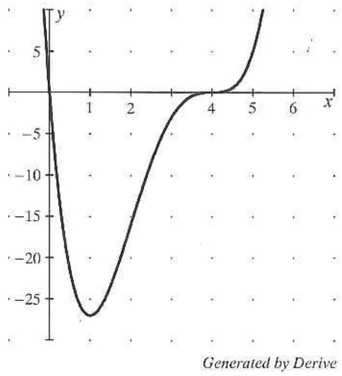
First derivative: $f'(x) = 4(x - 1)(x - 4)^2$
Second derivative: $f''(x) = 12(x - 4)(x - 2)$
x-intercepts: $(0, 0), (4, 0)$
y-intercept: $(0, 0)$
Vertical asymptotes: None
Horizontal asymptotes: None
Critical numbers: $x = 1, x = 4$
Possible points of inflection: $x = 2, x = 4$
Domain: All real numbers
Test intervals: $(-\infty, 1), (1, 2), (2, 4), (4, \infty)$

The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.49(a). Using a computer algebra system such as Derive (see Figure 3.49b) can help you verify your analysis.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 1$		$-$	$+$	Decreasing, concave up
$x = 1$	-27	0	$+$	Relative minimum
$1 < x < 2$		$+$	$+$	Increasing, concave up
$x = 2$	-16	$+$	0	Point of inflection
$2 < x < 4$		$+$	$-$	Increasing, concave down
$x = 4$	0	0	0	Point of inflection
$4 < x < \infty$		$+$	$+$	Increasing, concave up



(a)

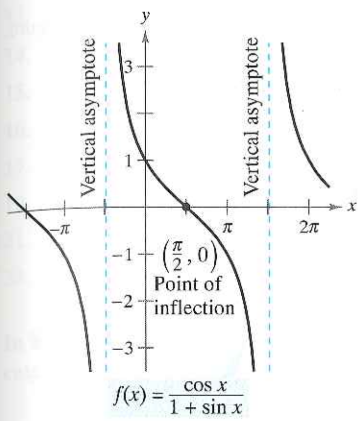


(b)

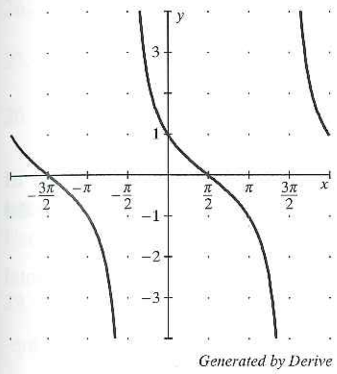
A polynomial function of even degree must have at least one relative extremum.
Figure 3.49

The fourth-degree polynomial function in Example 5 has one relative minimum and no relative maxima. In general, a polynomial function of degree n can have *at most* $n - 1$ relative extrema, and *at most* $n - 2$ points of inflection. Moreover, polynomial functions of even degree must have *at least* one relative extremum.

Remember from the leading coefficient test described in Section P.3 that the "end behavior" of the graph of a polynomial function is determined by its leading coefficient and its degree. For instance, because the polynomial in Example 5 has a positive leading coefficient, the graph moves up to the right. Moreover, because the degree is even, the graph also moves up to the left.



(a)



(b)

Figure 3.50

EXAMPLE 6 Analyzing the Graph of a Trigonometric Function

Analyze the graph of $f(x) = \dfrac{\cos x}{1 + \sin x}$.

Solution Because the function has a period of 2π, you can restrict the analysis of the graph to the interval $(-\pi/2, 3\pi/2)$.

First derivative: $f'(x) = -\dfrac{1}{1 + \sin x}$

Second derivative: $f''(x) = \dfrac{\cos x}{(1 + \sin x)^2}$

Period: 2π

x-intercept: $\left(\dfrac{\pi}{2}, 0\right)$

y-intercept: $(0, 1)$

Vertical asymptotes: $x = -\dfrac{\pi}{2}, x = \dfrac{3\pi}{2}$

Horizontal asymptotes: None

Critical numbers: None

Possible points of inflection: $x = \dfrac{\pi}{2}$

Domain: All real numbers except $x = \dfrac{3 + 4n}{2}\pi$

Test intervals: $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

The analysis of the graph of f on the interval $(-\pi/2, 3\pi/2)$ is shown in the table, and the graph is shown in Figure 3.50(a). Compare this with the graph generated by the computer algebra system Derive in Figure 3.50(b).

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$x = -\dfrac{\pi}{2}$	Undefined	Undefined	Undefined	Vertical asymptote
$-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$		$-$	$+$	Decreasing, concave up
$x = \dfrac{\pi}{2}$	0	$-\dfrac{1}{2}$	0	Point of inflection
$\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$		$-$	$-$	Decreasing, concave down
$x = \dfrac{3\pi}{2}$	Undefined	Undefined	Undefined	Vertical asymptote

NOTE The work involved in sketching the graph of a trigonometric function can be lessened sometimes by using trigonometric identities. For instance, the function in Example 6 can be rewritten as

$$f(x) = \frac{\cos x}{1 + \sin x} = \cot\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

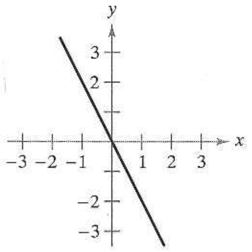
In this form, you can recognize the familiar cotangent graph shown in Figure 3.50.

EXERCISES FOR SECTION 3.6

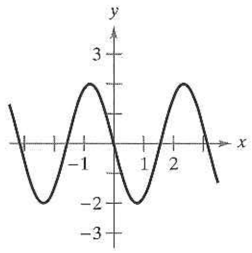
1. Match the graph of f in the left column with that of its derivative in the right column.

f | f'

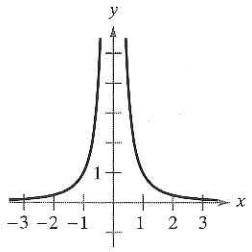
(a)



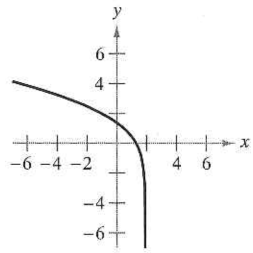
(A)



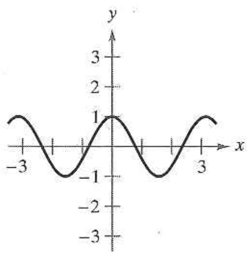
(b)



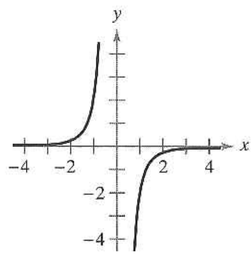
(B)



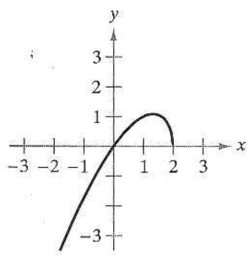
(c)



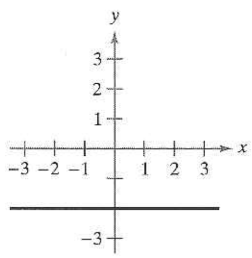
(C)



(d)



(D)

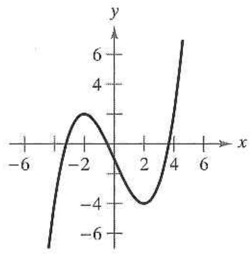


2. ***Think About It*** Suppose $f'(t) < 0$ for all t in the interval $(2, 8)$. Explain why $f(3) > f(5)$.

3. ***Think About It*** Suppose $f'(x) = \frac{2}{3}$ for all x and $f(0) = 1$. Find $f(6)$.

4. ***Think About It*** Suppose $f(0) = 3$ and $2 \le f'(x) \le 4$ for all x in the interval $[-5, 5]$. Determine the greatest and least possible values of $f(2)$.

5. ***Graphical Reasoning*** The graph of f is given in the figure.



(a) For which values of x is $f'(x)$ zero? Positive? Negative?

(b) For which values of x is $f''(x)$ zero? Positive? Negative?

(c) On what interval is f' an increasing function?

(d) For which value of x is $f'(x)$ minimum? For this value of x, how does the rate of change of f compare with the rate of change of f for other values of x? Explain.

6. ***Investigation*** Consider the function

$$f(x) = \frac{3x^n}{x^4 + 1}$$

for nonnegative integer values of n.

(a) Discuss the relationship between the value of n and the symmetry of the graph.

(b) For which values of n will the x-axis be the horizontal asymptote?

(c) For which value of n will $y = 3$ be the horizontal asymptote?

(d) What is the asymptote of the graph when $n = 5$?

(e) Use a graphing utility to graph f for the indicated values of n in the table. Use the graph to determine the number of extrema M and the number of inflection points N of the graph.

n	0	1	2	3	4	5
M						
N						

In Exercises 7–24, make use of domain, range, symmetry, asymptotes, intercepts, relative extrema, and/or points of inflection to sketch a graph of the function. You can use a graphing utility to verify your results.

7. $y = x^3 - 3x^2 + 3$

8. $y = -\frac{1}{3}(x^3 - 3x + 2)$

9. $y = 2 - x - x^3$

10. $f(x) = \frac{1}{3}(x - 1)^3 + 2$

11. $f(x) = 3x^3 - 9x + 1$

12. $f(x) = (x + 1)(x - 2)(x - 5)$

13. $y = 3x^4 + 4x^3$

14. $y = 3x^4 - 6x^2$

15. $f(x) = x^4 - 4x^3 + 16x$

16. $f(x) = x^4 - 8x^3 + 18x^2 - 16x + 5$

17. $y = x^5 - 5x$

18. $y = (x - 1)^5$

19. $y = |2x - 3|$

20. $y = |x^2 - 6x + 5|$

21. $y = x\sqrt{4 - x}$

22. $y = x\sqrt{4 - x^2}$

23. $y = 3x^{2/3} - 2x$

24. $y = 3x^{2/3} - x^2$

In Exercises 25–28, sketch a graph of the function over the indicated interval. Use a graphing utility to verify your graph.

Function	*Interval*
25. $y = \sin x - \frac{1}{18}\sin 3x$	$0 \le x \le 2\pi$
26. $y = \cos x - \frac{1}{2}\cos 2x$	$0 \le x \le 2\pi$
27. $y = 2x - \tan x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
28. $y = 2x + \cot x$	$0 < x < \pi$

In Exercises 29–40, sketch a graph of the function. Label any intercepts, relative extrema, points of inflection, or asymptotes. Use a graphing utility to verify your results.

29. $y = \dfrac{x^2}{x^2 + 3}$

30. $y = \dfrac{x}{x^2 + 1}$

31. $y = \dfrac{1}{x - 2} - 3$

32. $y = \dfrac{x^2 + 1}{x^2 - 2}$

33. $y = \dfrac{2x}{x^2 - 1}$

34. $f(x) = \dfrac{x + 2}{x}$

35. $g(x) = x + \dfrac{4}{x^2 + 1}$

36. $f(x) = x + \dfrac{32}{x^2}$

37. $f(x) = \dfrac{x^2 + 1}{x}$

38. $f(x) = \dfrac{x^3}{x^2 - 1}$

39. $y = \dfrac{x^2 - 6x + 12}{x - 4}$

40. $y = \dfrac{2x^2 - 5x + 5}{x - 2}$

In Exercises 41–44, use a symbolic differentiation utility to analyze and graph the function. Identify any relative extrema, points of inflection, and asymptotes.

41. $f(x) = \dfrac{20x}{x^2 + 1} - \dfrac{1}{x}$

42. $f(x) = 5\left(\dfrac{1}{x - 4} - \dfrac{1}{x + 2}\right)$

43. $f(x) = \dfrac{x}{\sqrt{x^2 + 7}}$

44. $f(x) = \dfrac{4x}{\sqrt{x^2 + 15}}$

In Exercises 45 and 46, use a graphing utility to graph the function. Use the graph to determine whether or not it is possible for the graph of a function to cross its horizontal asymptote. Do you think it is possible for the graph of a function to cross its vertical asymptote? Why or why not?

45. $f(x) = \dfrac{4(x - 1)^2}{x^2 - 4x + 5}$

46. $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

Writing **In Exercises 47 and 48, use a graphing utility to graph the function. Explain why there is no vertical asymptote when a superficial examination of the function may indicate that there should be one.**

47. $h(x) = \dfrac{6 - 2x}{3 - x}$

48. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

Writing **In Exercises 49 and 50, use a graphing utility to graph the function and determine the slant asymptote of the graph. Zoom out repeatedly and describe how the graph on the display appears to change. Why does this occur?**

49. $f(x) = -\dfrac{x^2 - 3x - 1}{x - 2}$

50. $g(x) = \dfrac{2x^2 - 8x - 15}{x - 5}$

Think About It **In Exercises 51–54, create a function whose graph has the indicated characteristics. (The answer is not unique.)**

51. Vertical asymptote: $x = 5$
    Horizontal asymptote: $y = 0$

52. Vertical asymptote: $x = -3$
    Horizontal asymptote: None

53. Vertical asymptote: $x = 5$
    Slant asymptote: $y = 3x + 2$

54. Vertical asymptote: $x = 0$
    Slant asymptote: $y = -x$

55. ***Graphical Reasoning*** Consider the function

$$f(x) = \frac{ax}{(x - b)^2}.$$

(a) Determine the effect on the graph of f if $b \neq 0$ and a is varied. Consider cases where a is positive and a is negative.

(b) Determine the effect on the graph of f if $a \neq 0$ and b is varied.

56. Consider the function

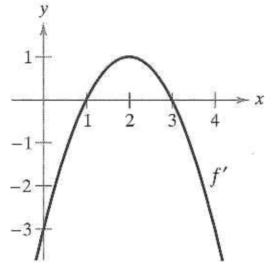
$$f(x) = \tfrac{1}{2}(ax)^2 - (ax), \quad a \neq 0.$$

(a) Determine the changes (if any) in the intercepts, extrema, and concavity of the graph of f when a is varied.

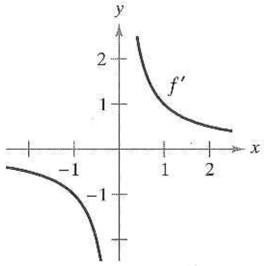
(b) In the same viewing rectangle, use a graphing utility to graph the function for four different values of a.

Think About It **In Exercises 57–62, use the graph (of f' or f'') to sketch a graph of the function f. (*Hint:* There is more than one correct answer.)**

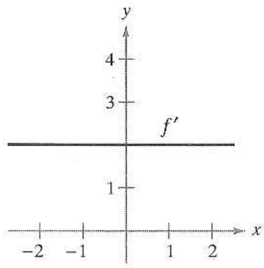
57.



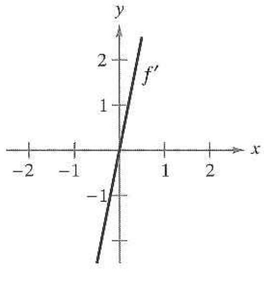
58.



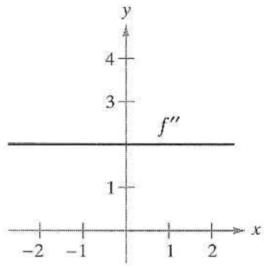
59.



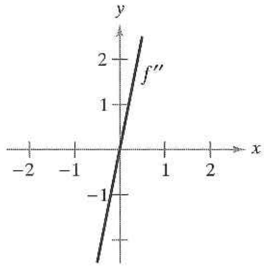
60.



61.

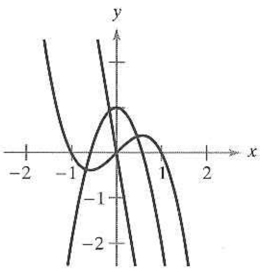


62.

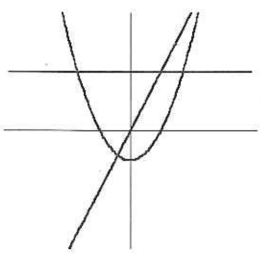


Think About It **In Exercises 63 and 64, the graphs of f, f', and f'' are shown on the same set of coordinate axes. Which is which?**

63.



64.



Generated by Maple

Think About It **In Exercises 65–70, determine conditions for the coefficients of**

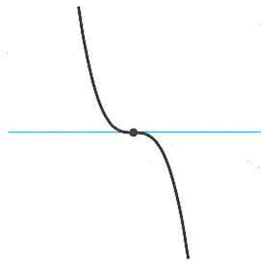
$$f(x) = ax^3 + bx^2 + cx + d$$

such that the graph of f resembles the given graph.

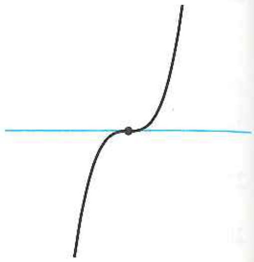
65.

66.

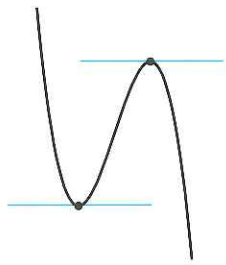
67.



68.



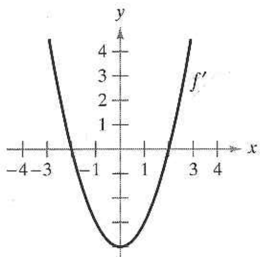
69.



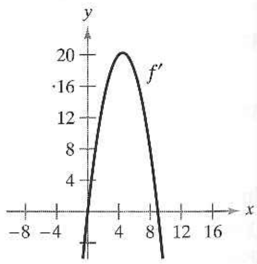
70.

In Exercises 71–74, use the graph of f' to sketch a graph of f and the graph of f''.

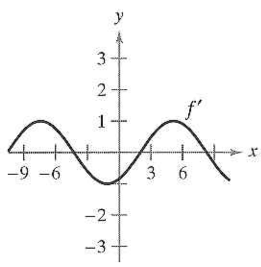
71.



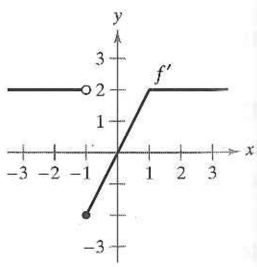
72.



73.



74.



(Submitted by Bill Fox, Moberly Area Community College, Moberly, MO.)