SECTION 3.4 Concavity and the Second Derivative Test

Concavity • Points of Inflection • The Second Derivative Test

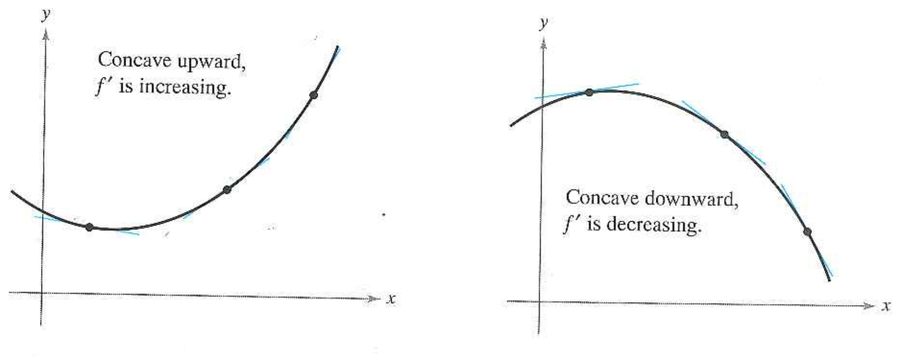
Concavity

You have already seen that locating the intervals in which a function f increases or decreases helps to describe its graph. In this section, you will see how locating the intervals in which f' increases or decreases can be used to determine where the graph of f is *curving upward* or *curving downward*.

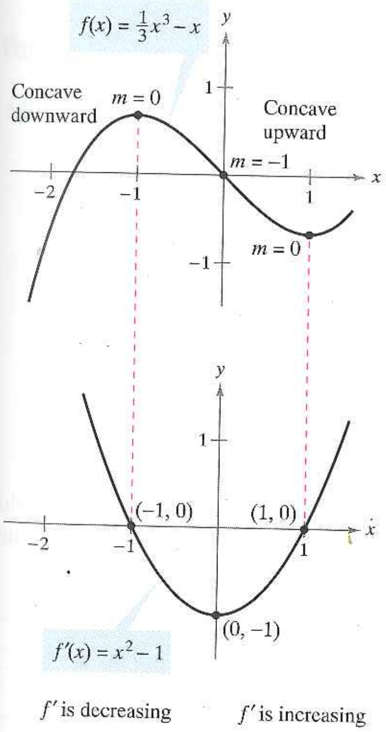
> **Definition of Concavity**
>
> Let f be differentiable on an open interval I. The graph of f is **concave upward** on I if f' is increasing on the interval and **concave downward** on I if f' is decreasing on the interval.

The following graphical interpretation of concavity is useful. (See the appendix for a proof of these results.)

1. Let f be differentiable at c. If the graph of f is concave upward at $(c, f(c))$, the graph of f lies *above* the tangent line at $(c, f(c))$ on some open interval containing c (see Figure 3.24a).
2. Let f be differentiable at c. If the graph of f is concave downward at $(c, f(c))$, the graph of f lies *below* the tangent line at $(c, f(c))$ on some open interval containing c (see Figure 3.24b).



(a) The graph of f lies above its tangent lines. (b) The graph of f lies below its tangent lines.
Figure 3.24



The concavity of f is related to the slope of the derivative.
Figure 3.25

To find the open intervals on which the graph of a function f is concave upward or downward, you need to find the intervals on which f' is increasing or decreasing. For instance, the graph of

$$f(x) = \frac{1}{3}x^3 - x$$

is concave downward on the open interval $(-\infty, 0)$ because $f'(x) = x^2 - 1$ is decreasing there. (See Figure 3.25.) Similarly, the graph of f is concave upward on the interval $(0, \infty)$ because f' is increasing on $(0, \infty)$.

The following theorem shows how to use the *second* derivative of a function f to determine intervals on which the graph of f is concave upward or downward. A proof of this theorem follows directly from Theorem 3.5 and the definition of concavity.

THEOREM 3.7 Test for Concavity

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward in I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward in I.

NOTE A third case of Theorem 3.7 could be that if $f''(x) = 0$ for all x in I, then f is linear. Note, however, that concavity is not defined for a line. In other words, a straight line is neither concave upward nor concave downward.

To apply Theorem 3.7, locate the x-values at which $f''(x) = 0$ or f'' is undefined. Second, use these x-values to determine test intervals. Finally, test the sign of $f''(x)$ in each of the test intervals.

EXAMPLE 1 Determining Concavity

Determine the open intervals on which the graph of

$$f(x) = 6(x^2 + 3)^{-1}$$

is concave upward or downward.

Solution Begin by observing that f is continuous on the entire real line. Next, find the second derivative of f.

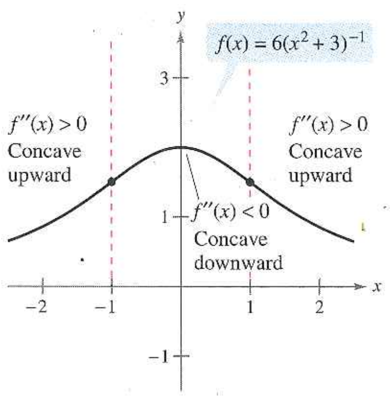
$$f(x) = 6(x^2 + 3)^{-1} \qquad \text{Original function}$$

$$f'(x) = (-6)(x^2 + 3)^{-2}(2x)$$

$$= \frac{-12x}{(x^2 + 3)^2} \qquad \text{First derivative}$$

$$f''(x) = \frac{(x^2 + 3)^2(-12) - (-12x)(2)(x^2 + 3)(2x)}{(x^2 + 3)^4}$$

$$= \frac{36(x^2 - 1)}{(x^2 + 3)^3} \qquad \text{Second derivative}$$



From the sign of f'' you can determine the concavity of the graph of f.
Figure 3.26

Because $f''(x) = 0$ when $x = \pm 1$ and f'' is defined on the entire real line, you should test f'' in the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. The results are shown in the table and in Figure 3.26.

Interval	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Test value	$x = -2$	$x = 0$	$x = 2$
Sign of $f''(x)$	$f''(-2) > 0$	$f''(0) < 0$	$f''(2) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

The function given in Example 1 is continuous on the entire real line. If there are x-values at which the function is not continuous, these values should be used along with the points at which $f''(x) = 0$ or is undefined to form the test intervals.

EXAMPLE 2 Determining Concavity

Determine the open intervals in which the graph of

$$f(x) = \frac{x^2 + 1}{x^2 - 4} \qquad \text{Original function}$$

is concave upward or downward.

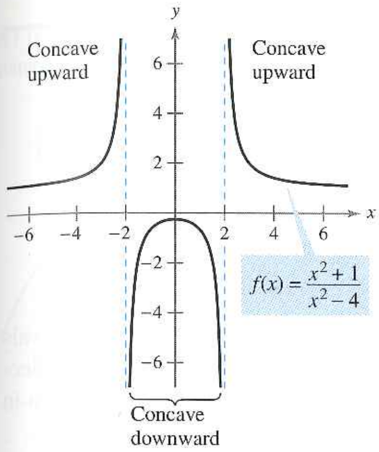


Figure 3.27

Solution

$$f'(x) = \frac{(x^2 - 4)(2x) - (x^2 + 1)(2x)}{(x^2 - 4)^2}$$

$$= \frac{-10x}{(x^2 - 4)^2} \qquad \text{First derivative}$$

$$f''(x) = \frac{(x^2 - 4)^2(-10) - (-10x)(2)(x^2 - 4)(2x)}{(x^2 - 4)^4}$$

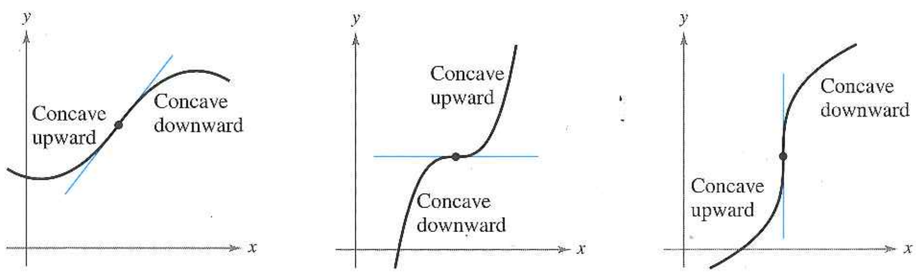
$$= \frac{10(3x^2 + 4)}{(x^2 - 4)^3} \qquad \text{Second derivative}$$

There are no points at which $f''(x) = 0$, but at $x = \pm 2$ the function f is not continuous, so you test for concavity in the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$, as shown in the table. The graph of f is shown in Figure 3.27.

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Test value	$x = -3$	$x = 0$	$x = 3$
Sign of $f''(x)$	$f''(-3) > 0$	$f''(0) < 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Points of Inflection

The graph in Figure 3.26 has two points at which the concavity changes. If the tangent line to the graph exists at such a point, that point is a **point of inflection.** Three types of points of inflection are shown in Figure 3.28. Note that a graph crosses its tangent line at a point of inflection.

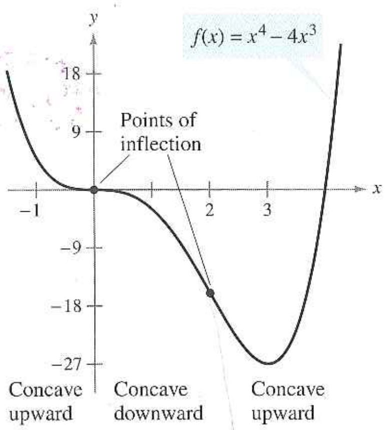


The concavity of f changes at a point of inflection.
Figure 3.28

To locate possible points of inflection, you need only determine the values of x for which $f''(x) = 0$ or for which f'' is undefined. This is similar to the procedure for locating relative extrema of f.

THEOREM 3.8 Points of Inflection

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f'' is undefined at $x = c$.



Points of inflection can occur where $f''(x) = 0$ or is undefined.
Figure 3.29

EXAMPLE 3 Finding Points of Inflection

Determine the points of inflection and discuss the concavity of the graph of

$f(x) = x^4 - 4x^3.$ Original function

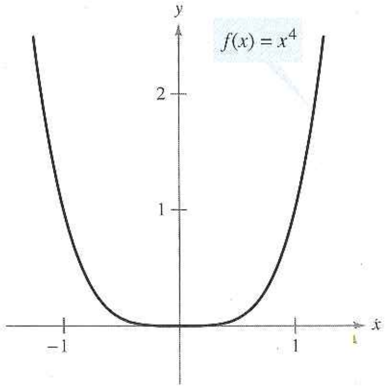
Solution Differentiating twice produces

$f'(x) = 4x^3 - 12x^2$ First derivative

$f''(x) = 12x^2 - 24x = 12x(x - 2).$ Second derivative

Possible points of inflection occur at $x = 0$ and $x = 2$. By testing the intervals determined by these x-values, you can conclude that they both yield points of inflection. A summary of this testing is shown in the table, and the graph of f is shown in Figure 3.29.

Interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test value	$x = -1$	$x = 1$	$x = 3$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(1) < 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave downward	Concave upward



$f''(0) = 0$, but $(0, 0)$ is *not* a point of inflection.
Figure 3.30

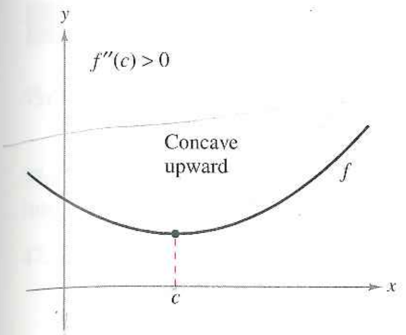
It is possible for the second derivative to be 0 at a point that is *not* a point of inflection. For instance, the graph of $f(x) = x^4$ is shown in Figure 3.30. The second derivative is 0 when $x = 0$, but the point $(0, 0)$ is not a point of inflection because the graph of f is concave upward in both intervals $-\infty < x < 0$ and $0 < x < \infty$.

EXPLORATION

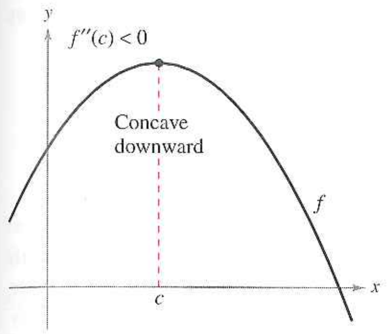
Consider a general cubic function of the form

$f(x) = ax^3 + bx^2 + cx + d.$

You know that the value of d has a bearing on the location of the graph but has no bearing on the value of the first derivative at given values of x. Graphically, this is true because changes in the value of d shift the graph up or down but do not change its basic shape. Use a graphing utility to graph several cubics with different values of c. Then give a graphical explanation of why changes in c do not affect the values of the second derivative.



If $f'(c) = 0$ and $f''(c) > 0$, $f(c)$ is a relative minimum.



If $f'(c) = 0$ and $f''(c) < 0$, $f(c)$ is a relative maximum.
Figure 3.31

The Second Derivative Test

In addition to testing for concavity, the second derivative can be used to perform a simple test for relative maxima and minima. The test is based on the fact that if the graph of a function f is concave upward on an open interval containing c, and $f'(c) = 0$, $f(c)$ must be a relative minimum of f. Similarly, if the graph of a function f is concave downward on an open interval containing c, and $f'(c) = 0$, $f(c)$ must be a relative maximum of f (see Figure 3.31).

THEOREM 3.9 Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.

If $f''(c) = 0$, the test fails. In such cases, you can use the First Derivative Test.

Proof If $f'(c) = 0$ and $f''(c) > 0$, there exists an open interval I containing c for which

$$\frac{f'(x) - f'(c)}{x - c} = \frac{f'(x)}{x - c} > 0$$

for all $x \neq c$ in I. If $x < c$, then $x - c < 0$ and $f'(x) < 0$. Also, if $x > c$, then $x - c > 0$ and $f'(x) > 0$. Thus, $f'(x)$ changes from negative to positive at c, and the First Derivative Test implies that $f(c)$ is a relative minimum. A proof of the second case is left to you.

EXAMPLE 4 Using the Second Derivative Test

Find the relative extrema for $f(x) = -3x^5 + 5x^3$.

Solution Begin by finding the critical numbers of f.

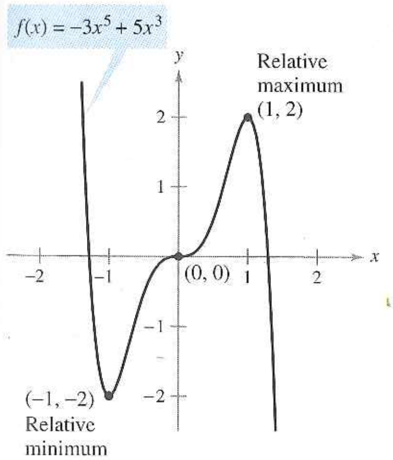
$$f'(x) = -15x^4 + 15x^2 = 15x^2(1 - x^2) = 0 \qquad \text{Set } f'(x) = 0.$$

$$x = -1, 0, 1 \qquad \text{Critical numbers}$$

Using $f''(x) = -60x^3 + 30x = 30(-2x^3 + x)$, you can apply the Second Derivative Test as follows.

Point	*Sign of f''*		*Conclusion*
$(-1, -2)$	$f''(-1) = 30 > 0$	⇨	Relative minimum
$(1, 2)$	$f''(1) = -30 < 0$	⇨	Relative maximum
$(0, 0)$	$f''(0) = 0$	⇨	Test fails

Because the Second Derivative Test fails at $(0, 0)$, you can use the First Derivative Test and observe that f increases to the left and right of $x = 0$. Thus, $(0, 0)$ is neither a relative minimum nor a relative maximum (even though the graph has a horizontal tangent line at this point). The graph of f is shown in Figure 3.32.



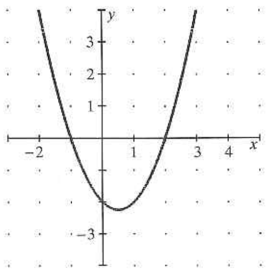
(0, 0) is neither a relative minimum nor a relative maximum.
Figure 3.32

1-6 7-19 21-29 41, 43, 45, 47, 51, 61

EXERCISES FOR SECTION 3.4

In Exercises 1–6, find the open intervals on which the graph is concave upward and those on which it is concave downward.

1. $y = x^2 - x - 2$



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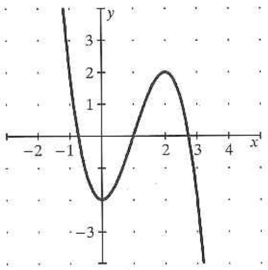
2. $y = -x^3 + 3x^2 - 2$



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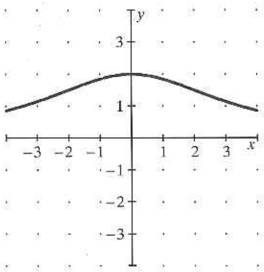
3. $f(x) = \dfrac{24}{x^2 + 12}$



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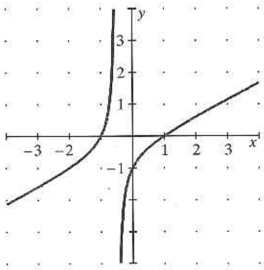
4. $f(x) = \dfrac{x^2 - 1}{2x + 1}$



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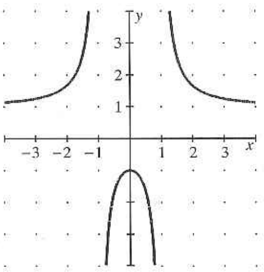
5. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$



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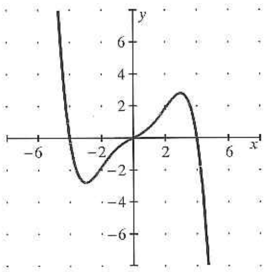
6. $y = \dfrac{-3x^5 + 40x^3 + 135x}{270}$



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In Exercises 7–20, find all relative extrema. Use the Second Derivative Test where applicable.

7. $f(x) = 6x - x^2$

8. $f(x) = x^2 + 3x - 8$

9. $f(x) = (x - 5)^2$

10. $f(x) = -(x - 5)^2$

11. $f(x) = x^3 - 3x^2 + 3$

12. $f(x) = 5 + 3x^2 - x^3$

13. $f(x) = x^4 - 4x^3 + 2$

14. $f(x) = x^3 - 9x^2 + 27x$

15. $f(x) = x^{2/3} - 3$

16. $f(x) = \sqrt{x^2 + 1}$

17. $f(x) = x + \dfrac{4}{x}$

18. $f(x) = \dfrac{x}{x - 1}$

19. $f(x) = \cos x - x$
$0 \le x \le 4\pi$

20. $f(x) = 2 \sin x + \cos 2x$
$0 \le x \le 2\pi$

In Exercises 21–36, find all relative extrema and points of inflection and use a graphing utility to graph the function.

21. $f(x) = x^3 - 12x$

22. $f(x) = x^3 + 1$

23. $f(x) = x^3 - 6x^2 + 12x$

24. $f(x) = 2x^3 - 3x^2 - 12x$

25. $f(x) = \dfrac{1}{4}x^4 - 2x^2$

26. $f(x) = 2x^4 - 8x + 3$

27. $f(x) = x(x - 4)^3$

28. $f(x) = x^3(x - 4)$

29. $f(x) = x\sqrt{x + 3}$

30. $f(x) = x\sqrt{x + 1}$

31. $f(x) = \sin \dfrac{x}{2}$
$0 \le x \le 4\pi$

32. $f(x) = 2 \csc \dfrac{3x}{2}$
$0 < x < 2\pi$

33. $f(x) = \sec\left(x - \dfrac{\pi}{2}\right)$
$0 < x < 4\pi$

34. $f(x) = \sin x + \cos x$
$0 \le x \le 2\pi$

35. $f(x) = 2 \sin x + \sin 2x$
$0 \le x \le 2\pi$

36. $f(x) = x - \sin x$
$0 \le x \le 4\pi$

In Exercises 37–40, use a symbolic differentiation utility to analyze the function over the indicated interval. (a) Find the first- and second-order derivatives of the function. (b) Find any relative extrema and points of inflection. (c) Graph f, f', and f'' on the same set of coordinate axes and state the relationship between the behavior of f and the signs of f' and f''.

37. $f(x) = 0.2x^2(x - 3)^3$, $[-1, 4]$

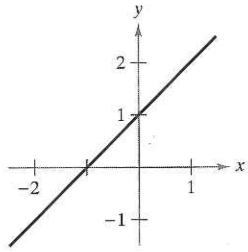
38. $f(x) = x^2\sqrt{6 - x^2}$, $\left[-\sqrt{6}, \sqrt{6}\right]$

39. $f(x) = \sin x - \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x$, $[0, \pi]$

40. $f(x) = \sqrt{2x} \sin x$, $[0, 2\pi]$

Think About It **In Exercises 41–44, trace the graph of f. On the same set of coordinate axes, sketch the graphs of f' and f''.**

41.



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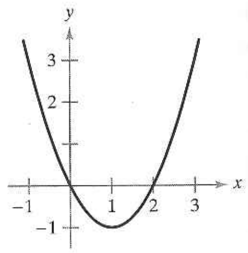

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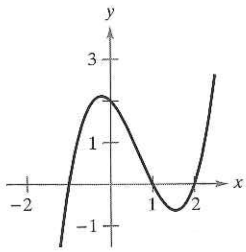

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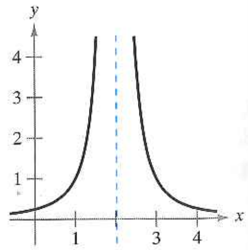


45. ***Think About It*** Consider a function f such that f' is increasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.

46. ***Think About It*** Consider a function f such that f' is decreasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.

Think About It **In Exercises 47–50, sketch the graph of a function f having the indicated characteristics.**

47. $f(2) = f(4) = 0$
$f(3)$ is defined.
$f'(x) < 0$ if $x < 3$
$f'(3)$ is undefined.
$f'(x) > 0$ if $x > 3$
$f''(x) < 0, x \neq 3$

48. $f(0) = f(2) = 0$
$f'(x) > 0$ if $x < 1$
$f'(1) = 0$
$f'(x) < 0$ if $x > 1$
$f''(x) < 0$

49. $f(2) = f(4) = 0$
$f'(x) > 0$ if $x < 3$
$f'(3)$ is undefined.
$f'(x) < 0$ if $x > 3$
$f''(x) > 0, x \neq 3$

50. $f(0) = f(2) = 0$
$f'(x) < 0$ if $x < 1$
$f'(1) = 0$
$f'(x) > 0$ if $x > 1$
$f''(x) > 0$

In Exercises 51 and 52, find *a, b, c,* and *d* such that the cubic $f(x) = ax^3 + bx^2 + cx + d$ satisfies the indicated conditions.

51. Relative maximum: $(3, 3)$
Relative minimum: $(5, 1)$
Inflection point: $(4, 2)$

52. Relative maximum: $(2, 4)$
Relative minimum: $(4, 2)$
Inflection point: $(3, 3)$

53. ***Think About It*** The figure shows the graph of the second derivative of a function f. Sketch a graph of f. (The answer is not unique.)

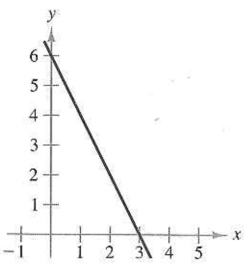


Figure for 53

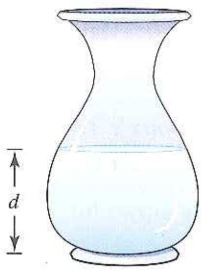


Figure for 54

54. ***Think About It*** Water is running into the vase shown in the figure at a constant rate.

(a) Sketch a graph of the depth d of water in the vase as a function of time.

(b) Does the function have any extrema? Explain.

(c) Give an interpretation of the inflection points of the graph of d.

55. ***Conjecture*** Consider the function $f(x) = (x - 2)^n$.

(a) Use a graphing utility to graph f for $n = 1, 2, 3$, and 4. Use the graphs to make a conjecture about the relationship between n and any inflection points of the graph of f.

(b) Verify your conjecture in part (a).

56. (a) Graph $f(x) = \sqrt[3]{x}$ and identify the inflection point.

(b) Does $f''(x)$ exist at the inflection point? Explain.

57. ***Think About It*** S represents weekly sales of a product. What can be said of S' and S'' for each of the following?

(a) The rate of change of sales is increasing.

(b) Sales are increasing at a slower rate.

(c) The rate of change of sales is constant.

(d) Sales are steady.

(e) Sales are declining, but at a slower rate.

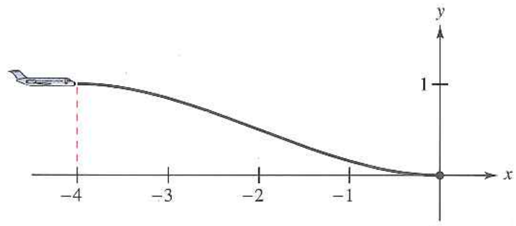
(f) Sales have bottomed out and have started to rise.

58. ***Think About It*** Sketch the graph of an arbitrary function that does *not* have a point of inflection at $(c, f(c))$ even though $f''(c) = 0$.

59. ***Aircraft Glide Path*** A small aircraft starts its descent from an altitude of 1 mile, 4 miles west of the runway (see figure).

(a) Find the cubic $f(x) = ax^3 + bx^2 + cx + d$ on the interval $[-4, 0]$ that describes a smooth glide path for the landing.

(b) If the glide path of the plane is described by the function in part (a), when would the plane be descending at the most rapid rate?



FOR FURTHER INFORMATION For more information on this type of modeling, see the article "How Not to Land at Lake Tahoe!" by Richard Barshinger in the May 1992 issue of the *The American Mathematical Monthly.*

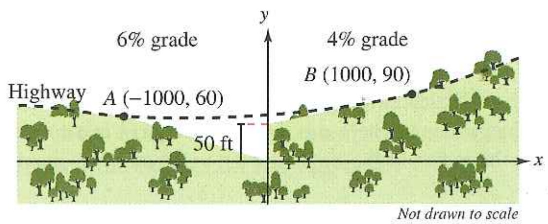
60. ***Highway Design*** A section of highway connecting two hillsides with grades of 6% and 4% is to be built between two points that are separated by a horizontal distance of 2000 feet (see figure). At the point where the two hillsides come together, there is a 50-foot difference in elevation.

(a) Design a section of highway connecting the hillsides modeled by the function $f(x) = ax^3 + bx^2 + cx + d\ (-1000 \leq x \leq 1000)$. At the points A and B, the slope of the model must match the grade of the hillside.

(b) Use a graphing utility to graph the model.

(c) Use a graphing utility to graph the derivative of the model.

(d) Determine the grade at the steepest part of the transitional section of the highway.



61. ***Beam Deflection*** The deflection D of a particular beam of length L is

$$D = 2x^4 - 5Lx^3 + 3L^2x^2$$

where x is the distance from one end of the beam. Find the value of x that yields the maximum deflection.

62. ***Specific Gravity*** A model for the specific gravity of water S is

$$S = \frac{5.755}{10^8}T^3 - \frac{8.521}{10^6}T^2 + \frac{6.540}{10^5}T + 0.99987, \quad 0 < T < 25$$

where T is the water temperature in degrees Celsius.

(a) Use a symbolic differentiation utility to find the coordinates of the maximum value of the function.

(b) Sketch a graph of the function over the specified domain. (Use a setting in which $0.996 \le S \le 1.001$.)

(c) Estimate the specific gravity of water when $T = 20°$.

63. ***Average Cost*** A manufacturer has determined that the total cost C of operating a factory is

$$C = 0.5x^2 + 15x + 5000$$

where x is the number of units produced. At what level of production will the average cost per unit be minimized? (The average cost per unit is C/x.)

64. ***Inventory Cost*** The total cost C for ordering and storing x units is

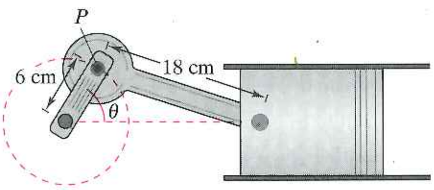
$$C = 2x + \frac{300{,}000}{x}.$$

What order size will produce a minimum cost?

65. ***Engine Design*** In the engine shown in the figure, a connecting rod 18 centimeters long is fastened to a crank of radius 6 centimeters at point P. The crankshaft rotates counterclockwise at a constant rate of 200 revolutions per minute. The horizontal velocity (cm/min) of point P is

$$v = -2400\pi \sin\theta$$

where θ is the central angle of the crankshaft. What values of θ produce a maximum horizontal velocity?



66. ***Electric Field Intensity*** The equation

$$E = \frac{kqx}{(x^2 + a^2)^{3/2}}$$

gives the electric field intensity on the axis of a uniformly charged ring, where q is the total charge, k is a constant, and a is the radius of the ring. At what value of x is E maximum?

Linear and Quadratic Approximations **In Exercises 67–70, use a graphing utility to graph the function. Then graph the linear and quadratic approximations**

$$P_1(x) = f(a) + f'(a)(x - a)$$

and

$$P_2(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2$$

in the same viewing rectangle. Compare the values of f, P_1, and P_2 and their first derivatives at $x = a$. How do the approximations change as you move away from $x = a$?

Function	*Value of a*
67. $f(x) = 2(\sin x + \cos x)$	$a = 0$
68. $f(x) = \sqrt{1 - x}$	$a = 0$
69. $f(x) = 2(\sin x + \cos x)$	$a = \frac{\pi}{4}$
70. $f(x) = \frac{\sqrt{x}}{x - 1}$	$a = 2$

71. Use a graphing utility to graph $y = x\sin(1/x)$. Show that the graph is concave downward to the right of $x = 1/\pi$.

72. Show that the point of inflection of $f(x) = x(x - 6)^2$ lies midway between the relative extrema of f.

73. Prove that every cubic function with three distinct real zeros has a point of inflection whose x-coordinate is the average of the three zeros.

74. Show that the cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ has exactly one point of inflection (x_0, y_0), where

$$x_0 = \frac{-b}{3a} \quad \text{and} \quad y_0 = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d.$$

Use this formula to find the point of inflection of $p(x) = x^3 - 3x^2 + 2$.

75. ***Darboux's Theorem*** Prove Darboux's Theorem: Let $f(x)$ be differentiable on $[a, b]$, $f'(a) = y_1$, and $f'(b) = y_2$. If d lies between y_1 and y_2, then there exists c in (a, b) such that $f'(c) = d$.

76. ***Writing*** Discuss the advantages and disadvantages of the First and Second Derivative Tests. Illustrate your discussion with examples.

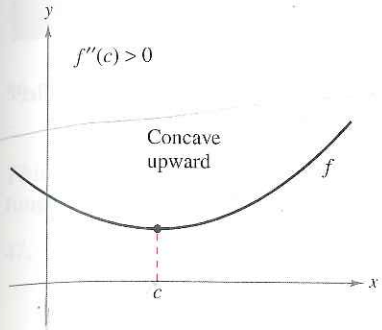
True or False? **In Exercises 77–80, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

77. The graph of every cubic polynomial has precisely one point of inflection.

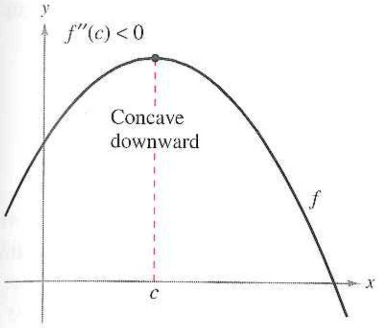
78. The graph of $f(x) = 1/x$ is concave downward for $x < 0$ and concave upward for $x > 0$, and thus it has a point of inflection at $x = 0$.

79. The maximum value of $y = 3\sin x + 2\cos x$ is 5.

80. The maximum slope of the graph of $y = \sin(bx)$ is b.



If $f'(c) = 0$ and $f''(c) > 0$, $f(c)$ is a relative minimum.



If $f'(c) = 0$ and $f''(c) < 0$, $f(c)$ is a relative maximum.
Figure 3.31

The Second Derivative Test

In addition to testing for concavity, the second derivative can be used to perform a simple test for relative maxima and minima. The test is based on the fact that if the graph of a function f is concave upward on an open interval containing c, and $f'(c) = 0$, $f(c)$ must be a relative minimum of f. Similarly, if the graph of a function f is concave downward on an open interval containing c, and $f'(c) = 0$, $f(c)$ must be a relative maximum of f (see Figure 3.31).

THEOREM 3.9 Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.

If $f''(c) = 0$, the test fails. In such cases, you can use the First Derivative Test.

Proof If $f'(c) = 0$ and $f''(c) > 0$, there exists an open interval I containing c for which

$$\frac{f'(x) - f'(c)}{x - c} = \frac{f'(x)}{x - c} > 0$$

for all $x \neq c$ in I. If $x < c$, then $x - c < 0$ and $f'(x) < 0$. Also, if $x > c$, then $x - c > 0$ and $f'(x) > 0$. Thus, $f'(x)$ changes from negative to positive at c, and the First Derivative Test implies that $f(c)$ is a relative minimum. A proof of the second case is left to you.

EXAMPLE 4 Using the Second Derivative Test

Find the relative extrema for $f(x) = -3x^5 + 5x^3$.

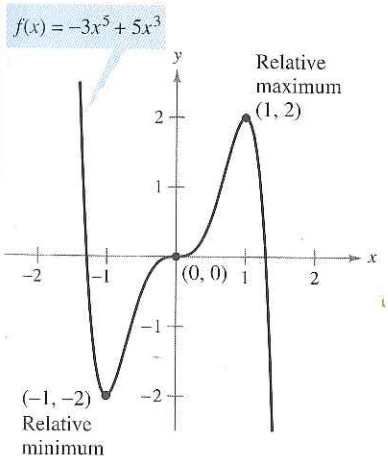
Solution Begin by finding the critical numbers of f.

$$f'(x) = -15x^4 + 15x^2 = 15x^2(1 - x^2) = 0 \qquad \text{Set } f'(x) = 0.$$
$$x = -1, 0, 1 \qquad \text{Critical numbers}$$

Using $f''(x) = -60x^3 + 30x = 30(-2x^3 + x)$, you can apply the Second Derivative Test as follows.

Point	*Sign of f''*		*Conclusion*
$(-1, -2)$	$f''(-1) = 30 > 0$	⇨	Relative minimum
$(1, 2)$	$f''(1) = -30 < 0$	⇨	Relative maximum
$(0, 0)$	$f''(0) = 0$	⇨	Test fails

Because the Second Derivative Test fails at $(0, 0)$, you can use the First Derivative Test and observe that f increases to the left and right of $x = 0$. Thus, $(0, 0)$ is neither a relative minimum nor a relative maximum (even though the graph has a horizontal tangent line at this point). The graph of f is shown in Figure 3.32.



(0, 0) is neither a relative minimum nor a relative maximum.
Figure 3.32