SECTION 2.4 The Chain Rule

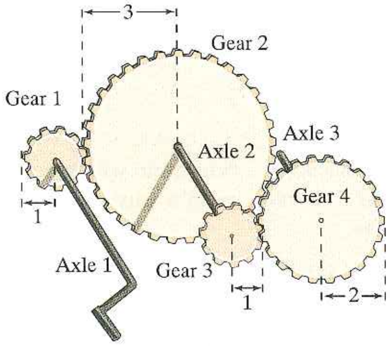
The Chain Rule • The General Power Rule • Simplifying Derivatives • Trigonometric Functions and the Chain Rule

The Chain Rule

We have yet to discuss one of the most powerful differentiation rules—the **Chain Rule.** This rule deals with composite functions and adds a surprising versatility to the rules discussed in the two previous sections. For example, compare the following functions. Those on the left can be differentiated without the Chain Rule, and those on the right are best done with the Chain Rule.

Without the Chain Rule	*With the Chain Rule*
$y = x^2 + 1$	$y = \sqrt{x^2 + 1}$
$y = \sin x$	$y = \sin 6x$
$y = 3x + 2$	$y = (3x + 2)^5$
$y = x + \tan x$	$y = x + \tan x^2$

Basically, the Chain Rule states that if y changes dy/du times as fast as u, and u changes du/dx times as fast as x, then y changes $(dy/du)(du/dx)$ times as fast as x.



Axle 1: y revolutions per minute
Axle 2: u revolutions per minute
Axle 3: x revolutions per minute
Figure 2.23

EXAMPLE 1 The Derivative of a Composite Function

A set of gears is constructed, as shown in Figure 2.23, such that the second and third gears are on the same axle. As the first axle revolves, it drives the second axle, which in turn drives the third axle. Let y, u, and x represent the numbers of revolutions per minute of the first, second, and third axles. Find dy/du, du/dx, and dy/dx, and show that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Solution Because the circumference of the second gear is three times that of the first, the first axle must make three revolutions to turn the second axle once. Similarly, the second axle must make two revolutions to turn the third axle once, and you can write

$$\frac{dy}{du} = 3 \quad \text{and} \quad \frac{du}{dx} = 2.$$

Combining these two results, you know that the first axle must make six revolutions to turn the third axle once. Thus, you can write

$$\frac{dy}{dx} = \boxed{\text{Rate of change of first axle with respect to second axle}} \cdot \boxed{\text{Rate of change of second axle with respect to third axle}}$$

$$= \frac{dy}{du} \cdot \frac{du}{dx} = 3 \cdot 2 = 6$$

$$= \boxed{\text{Rate of change of first axle with respect to third axle}}.$$

In other words, the rate of change of y with respect to x is the product of the rate of change of y with respect to u and the rate of change of u with respect to x.

EXPLORATION

Using the Chain Rule Each of the following functions can be differentiated using rules that you studied in Sections 2.2 and 2.3. For each function, find the derivative using those rules. Then find the derivative using the Chain Rule. Compare your results. Which method is simpler?

(a) $\dfrac{2}{3x + 1}$

(b) $(x + 2)^3$

(c) $\sin 2x$

Example 1 illustrates a simple case of the Chain Rule. The general rule is stated below.

THEOREM 2.10 The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

Proof Let $h(x) = f(g(x))$. Then, using the alternative form of the derivative, you need to show that, for $x = c$,

$$h'(c) = f'(g(c))g'(c).$$

An important consideration in this proof is the behavior of g as x approaches c. A problem occurs if there are values of x, other than c, such that $g(x) = g(c)$. In the appendix we show how to use the differentiability of f and g to overcome this problem. For now, assume that $g(x) \neq g(c)$ for values of x other than c. In the proofs of the Product Rule and the Quotient Rule, we added and subtracted the same quantity to obtain the desired form. This proof uses a similar technique—multiplying and dividing by the same (nonzero) quantity. Note that because g is differentiable, it is also continuous, and it follows that $g(x) \to g(c)$ as $x \to c$.

$$\begin{aligned}
h'(c) &= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} \\
&= \lim_{x \to c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c}\right], \quad g(x) \neq g(c) \\
&= \left[\lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)}\right]\left[\lim_{x \to c} \frac{g(x) - g(c)}{x - c}\right] \\
&= f'(g(c))g'(c)
\end{aligned}$$

When applying the Chain Rule, it is helpful to think of the composite function $f \circ g$ as having two parts—an inner part and an outer part.

Outer function

$$y = f(g(x)) = f(u)$$

Inner function

The derivative of $y = f(u)$ is the derivative of the outer function (at the inner function u) *times* the derivative of the inner function.

$$y' = f'(u) \cdot u'$$

EXAMPLE 2 Decomposition of a Composite Function

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
a. $y = \dfrac{1}{x+1}$	$u = x + 1$	$y = \dfrac{1}{u}$
b. $y = \sin 2x$	$u = 2x$	$y = \sin u$
c. $y = \sqrt{3x^2 - x + 1}$	$u = 3x^2 - x + 1$	$y = \sqrt{u}$
d. $y = \tan^2 x$	$u = \tan x$	$y = u^2$

EXAMPLE 3 Using the Chain Rule

Find dy/dx for $y = (x^2 + 1)^3$.

Solution For this function, you can consider the inside function to be $u = x^2 + 1$. By the Chain Rule, you obtain

$$\frac{dy}{dx} = \underbrace{3(x^2+1)^2}_{\frac{dy}{du}}\underbrace{(2x)}_{\frac{du}{dx}} = 6x(x^2+1)^2.$$

STUDY TIP You could also solve the problem in Example 3 without using the Chain Rule by observing that

$$y = x^6 + 3x^4 + 3x^2 + 1$$

and

$$y' = 6x^5 + 12x^3 + 6x.$$

Verify that this is the same as the derivative in Example 3. Which method would you use to find

$$\frac{d}{dx}(x^2 + 1)^{50}?$$

The General Power Rule

The function in Example 3 is an example of one of the most common types of composite functions, $y = [u(x)]^n$. The rule for differentiating such functions is called the **General Power Rule,** and it is a special case of the Chain Rule.

THEOREM 2.11 The General Power Rule

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = nu^{n-1}u'.$$

Proof Because $y = u^n$, you apply the Chain Rule to obtain

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$
$$= \frac{d}{du}[u^n]\frac{du}{dx}.$$

By the (simple) Power Rule in Section 2.2, you have $D_u[u^n] = nu^{n-1}$, and it follows that $dy/dx = n[u(x)]^{n-1}(du/dx)$.

EXAMPLE 4 Applying the General Power Rule

Find the derivative of $f(x) = (3x - 2x^2)^3$.

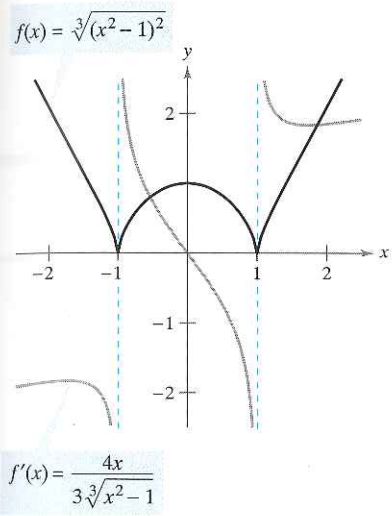
Solution Let $u = 3x - 2x^2$. Then

$$f(x) = (3x - 2x^2)^3 = u^3$$

and, by the General Power Rule, the derivative is

$$f'(x) = \underbrace{3}_{n}\underbrace{(3x - 2x^2)^2}_{u^{n-1}}\underbrace{\frac{d}{dx}[3x - 2x^2]}_{u'}$$ Apply General Power Rule.

$$= 3(3x - 2x^2)^2(3 - 4x).$$ Differentiate $3x - 2x^2$.



The derivative of f is 0 at $x = 0$ and is undefined at $x = \pm 1$.
Figure 2.24

EXAMPLE 5 Differentiating Functions Involving Radicals

Find all points on the graph of $f(x) = \sqrt[3]{(x^2 - 1)^2}$ for which $f'(x) = 0$ and those for which $f'(x)$ does not exist.

Solution Begin by rewriting the function as

$$f(x) = (x^2 - 1)^{2/3}.$$

Then, applying the General Power Rule (with $u = x^2 - 1$) produces

$$f'(x) = \underbrace{\frac{2}{3}}_{n}\underbrace{(x^2 - 1)^{-1/3}}_{u^{n-1}}\underbrace{(2x)}_{u'}$$ Apply General Power Rule.

$$= \frac{4x}{3\sqrt[3]{x^2 - 1}}.$$ Write in radical form.

Thus, $f'(x) = 0$ when $x = 0$ and $f'(x)$ does not exist when $x = \pm 1$, as indicated in Figure 2.24.

EXAMPLE 6 Differentiating Quotients with Constant Numerators

Differentiate $g(t) = \frac{-7}{(2t - 3)^2}$.

Solution Begin by rewriting the function as

$$g(t) = -7(2t - 3)^{-2}.$$

Then, applying the General Power Rule produces

$$g'(t) = \underbrace{(-7)}_{\text{Constant Multiple Rule}}\underbrace{(-2)}_{n}\underbrace{(2t - 3)^{-3}}_{u^{n-1}}\underbrace{(2)}_{u'}$$ Apply General Power Rule.

$$= 28(2t - 3)^{-3}$$ Simplify.

$$= \frac{28}{(2t - 3)^3}.$$ Write with positive exponent.

NOTE Try differentiating the function in Example 6 using the Quotient Rule. You should obtain the same result, but using the Quotient Rule is less efficient than using the General Power Rule.

Simplifying Derivatives

The next three examples illustrate some techniques for simplifying the "raw derivatives" of functions involving products, quotients, and composites.

EXAMPLE 7 Simplifying by Factoring Out the Least Powers

$$f(x) = x^2\sqrt{1-x^2} \qquad \text{Original function}$$

$$= x^2(1-x^2)^{1/2} \qquad \text{Rewrite.}$$

$$f'(x) = x^2\frac{d}{dx}[(1-x^2)^{1/2}] + (1-x^2)^{1/2}\frac{d}{dx}[x^2] \qquad \text{Product Rule}$$

$$= x^2\left[\frac{1}{2}(1-x^2)^{-1/2}(-2x)\right] + (1-x^2)^{1/2}(2x) \qquad \text{General Power Rule}$$

$$= -x^3(1-x^2)^{-1/2} + 2x(1-x^2)^{1/2} \qquad \text{Simplify.}$$

$$= x(1-x^2)^{-1/2}[-x^2(1) + 2(1-x^2)] \qquad \text{Factor.}$$

$$= \frac{x(2-3x^2)}{\sqrt{1-x^2}} \qquad \text{Simplify.}$$

EXAMPLE 8 Simplifying the Derivative of a Quotient

$$f(x) = \frac{x}{\sqrt[3]{x^2+4}} \qquad \text{Original function}$$

$$= \frac{x}{(x^2+4)^{1/3}} \qquad \text{Rewrite.}$$

$$f'(x) = \frac{(x^2+4)^{1/3}(1) - x(1/3)(x^2+4)^{-2/3}(2x)}{(x^2+4)^{2/3}} \qquad \text{Quotient Rule}$$

$$= \frac{1}{3}(x^2+4)^{-2/3}\left[\frac{3(x^2+4) - (2x^2)(1)}{(x^2+4)^{2/3}}\right] \qquad \text{Factor.}$$

$$= \frac{x^2+12}{3(x^2+4)^{4/3}} \qquad \text{Simplify.}$$

TECHNOLOGY Symbolic differentiation utilities are capable of differentiating very complicated functions. Often, however, the result is given in unsimplified form. If you have access to such a utility, use it to find the derivatives of the functions given in Examples 7, 8, and 9. Then compare the results with those given on this page.

EXAMPLE 9 Simplifying the Derivative of a Power

$$y = \left(\frac{3x-1}{x^2+3}\right)^2 \qquad \text{Original function}$$

$$y' = \underbrace{2}_{n}\underbrace{\left(\frac{3x-1}{x^2+3}\right)}_{u^{n-1}}\underbrace{\frac{d}{dx}\left[\frac{3x-1}{x^2+3}\right]}_{u'} \qquad \text{General Power Rule}$$

$$= \left[\frac{2(3x-1)}{x^2+3}\right]\left[\frac{(x^2+3)(3)-(3x-1)(2x)}{(x^2+3)^2}\right] \qquad \text{Quotient Rule}$$

$$= \frac{2(3x-1)(3x^2+9-6x^2+2x)}{(x^2+3)^3} \qquad \text{Multiply.}$$

$$= \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3} \qquad \text{Simplify.}$$

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Trigonometric Functions and the Chain Rule

The "Chain Rule versions" of the derivatives of the six trigonometric functions are as follows.

$$\frac{d}{dx}[\sin u] = (\cos u)\,u' \qquad \frac{d}{dx}[\cos u] = -(\sin u)\,u'$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u)\,u' \qquad \frac{d}{dx}[\cot u] = -(\csc^2 u)\,u'$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u)\,u' \qquad \frac{d}{dx}[\csc u] = -(\csc u \cot u)\,u'$$

EXAMPLE 10 Applying the Chain Rule to Trigonometric Functions

a. $y = \sin \overbrace{2x}^{u}$ $\qquad y' = \overbrace{\cos 2x}^{\cos u}\, \overbrace{\frac{d}{dx}[2x]}^{u'} = (\cos 2x)(2) = 2\cos 2x$

b. $y = \cos(x - 1)$ $\qquad y' = -\sin(x - 1)$

c. $y = \tan 3x$ $\qquad y' = 3\sec^2 3x$

Be sure that you understand the mathematical conventions regarding parentheses and trigonometric functions. For instance, in Example 10a, $\sin 2x$ is written to mean $\sin(2x)$.

EXAMPLE 11 Parentheses and Trigonometric Functions

a. $y = \cos 3x^2 = \cos(3x^2)$ $\qquad y' = (-\sin 3x^2)(6x) = -6x \sin 3x^2$

b. $y = (\cos 3)x^2$ $\qquad y' = (\cos 3)(2x) = 2x \cos 3$

c. $y = \cos(3x)^2 = \cos(9x^2)$ $\qquad y' = (-\sin 9x^2)(18x) = -18x \sin 9x^2$

d. $y = \cos^2 x = (\cos x)^2$ $\qquad y' = 2(\cos x)(-\sin x)$

$\qquad = -2\cos x \sin x$

To find the derivative of a function of the form $k(x) = f(g(h(x)))$, you need to apply the Chain Rule twice, as shown in Example 12.

EXAMPLE 12 Repeated Application of the Chain Rule

$f(t) = \sin^3 4t$ — Original function

$= (\sin 4t)^3$ — Rewrite.

$f'(t) = 3(\sin 4t)^2 \frac{d}{dt}[\sin 4t]$ — Apply Chain Rule once.

$= 3(\sin 4t)^2(\cos 4t)\frac{d}{dt}[4t]$ — Apply Chain Rule a second time.

$= 3(\sin 4t)^2(\cos 4t)(4)$

$= 12\sin^2 4t \cos 4t$ — Simplify.

We conclude this section with a summary of the differentiation rules studied so far. To become skilled at differentiation, you should memorize each rule.

Summary of Differentiation Rules

General Differentiation Rules

Let u and v be differentiable functions of x.

Constant Multiple Rule:

$$\frac{d}{dx}[cu] = cu'$$

Sum or Difference Rule:

$$\frac{d}{dx}[u \pm v] = u' \pm v'$$

Product Rule:

$$\frac{d}{dx}[uv] = uv' + vu'$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$$

Derivatives of Algebraic Functions

Constant Rule:

$$\frac{d}{dx}[c] = 0$$

Simple Power Rule:

$$\frac{d}{dx}[x^n] = nx^{n-1}, \quad \frac{d}{dx}[x] = 1$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

Chain Rule

Chain Rule:

$$\frac{d}{dx}[f(u)] = f'(u)\,u'$$

General Power Rule:

$$\frac{d}{dx}[u^n] = nu^{n-1}\,u'$$

STUDY TIP As an aid to memorization, note that the cofunctions (cosine, cotangent, and cosecant) require a negative sign as part of their derivatives.

EXERCISES FOR SECTION 2.4

In Exercises 1–6, complete the table using Example 2 as a model.

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
1. $y = (6x - 5)^4$		
2. $y = \dfrac{1}{\sqrt{x+1}}$		
3. $y = \sqrt{x^2 - 1}$		
4. $y = \tan(\pi x + 1)$		
5. $y = \csc^3 x$		
6. $y = \cos \dfrac{3x}{2}$		

In Exercises 7–30, find the first derivative of the algebraic function.

7. $y = (2x - 7)^3$
8. $y = (3x^2 + 1)^4$
9. $g(x) = 3(4 - 9x)^4$
10. $f(x) = 2(1 - x^2)^3$
11. $f(x) = (9 - x^2)^{2/3}$
12. $f(t) = (9t + 2)^{2/3}$
13. $f(t) = \sqrt{1 - t}$
14. $g(x) = \sqrt{3 - 2x}$
15. $y = \sqrt[3]{9x^2 + 4}$
16. $g(x) = \sqrt{x^2 - 2x + 1}$
17. $y = 2\sqrt{4 - x^2}$
18. $f(x) = -3\sqrt[4]{2 - 9x}$
19. $y = \dfrac{1}{x - 2}$
20. $s(t) = \dfrac{1}{t^2 + 3t - 1}$
21. $f(t) = \left(\dfrac{1}{t - 3}\right)^2$
22. $y = -\dfrac{4}{(t + 2)^2}$

23. $y = \dfrac{1}{\sqrt{x+2}}$

24. $g(t) = \sqrt{\dfrac{1}{t^2 - 2}}$

25. $f(x) = x^2(x-2)^4$

26. $f(x) = x(3x-9)^3$

27. $y = x\sqrt{1-x^2}$

28. $y = x^2\sqrt{9-x^2}$

29. $y = \dfrac{x}{\sqrt{x^2+1}}$

30. $y = \dfrac{x^2}{\sqrt{x^9+9}}$

In Exercises 31–40, use a symbolic differentiation utility to find the first derivative of the function. Then use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.

31. $y = \dfrac{\sqrt{x}+1}{x^2+1}$

32. $y = \sqrt{\dfrac{2x}{x+1}}$

33. $g(t) = \dfrac{3t^2}{\sqrt{t^2+2t-1}}$

34. $f(x) = \sqrt{x}(2-x)^2$

35. $y = \sqrt{\dfrac{x+1}{x}}$

36. $y = (t^2-9)\sqrt{t+2}$

37. $s(t) = \dfrac{-2(2-t)\sqrt{1+t}}{3}$

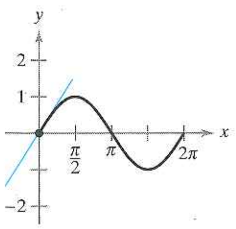
38. $g(x) = \sqrt{x-1} + \sqrt{x+1}$

39. $y = \dfrac{\cos \pi x + 1}{x}$

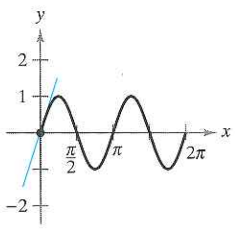
40. $y = x^2 \tan \dfrac{1}{x}$

In Exercises 41 and 42, find the slope of the tangent line to the sine function at the origin. Compare this value with the number of complete cycles in the interval $[0, 2\pi]$.

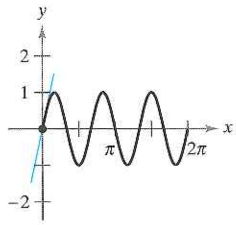
41. (a) $y = \sin x$



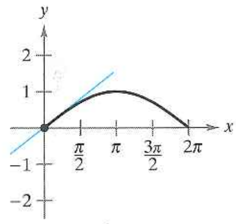
(b) $y = \sin 2x$



42. (a) $y = \sin 3x$



(b) $y = \sin \dfrac{x}{2}$



In Exercises 43–52, find the first derivative of the function.

43. $y = \cos 3x$

44. $y = \sin \pi x$

45. $g(x) = 3 \tan 4x$

46. $h(x) = \sec x^2$

47. $f(\theta) = \frac{1}{4}\sin^2 2\theta$

48. $g(t) = 5\cos^2 \pi t$

49. $y = \sqrt{x} + \frac{1}{4}\sin(2x)^2$

50. $y = 3x - 5\cos(\pi x)^2$

51. $y = \sin(\cos x)$

52. $y = \sin\sqrt{x} + \sqrt{\sin x}$

In Exercises 53–60, evaluate the derivative of the function at the indicated point. You can use a graphing utility to verify your result.

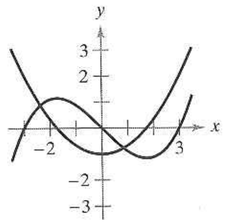
	Function	*Point*
53.	$s(t) = \sqrt{t^2+2t+8}$	$(2, 4)$
54.	$y = \sqrt[5]{3x^3+4x}$	$(2, 2)$
55.	$f(x) = \dfrac{3}{x^3-4}$	$\left(-1, -\dfrac{3}{5}\right)$
56.	$f(x) = \dfrac{1}{(x^2-3x)^2}$	$\left(4, \dfrac{1}{16}\right)$
57.	$f(t) = \dfrac{3t+2}{t-1}$	$(0, -2)$
58.	$f(x) = \dfrac{x+1}{2x-3}$	$(2, 3)$
59.	$y = 37 - \sec^3(2x)$	$(0, 36)$
60.	$y = \dfrac{1}{x} + \sqrt{\cos x}$	$\left(\dfrac{\pi}{2}, \dfrac{2}{\pi}\right)$

In Exercises 61–64, (a) find an equation of the tangent line to the graph of f at the indicated point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the derivative feature of a graphing utility to confirm your results.

	Function	*Point*
61.	$f(x) = \sqrt{3x^2-2}$	$(3, 5)$
62.	$f(x) = \frac{1}{3}x\sqrt{x^2+5}$	$(2, 2)$
63.	$f(x) = \sin 2x$	$(\pi, 0)$
64.	$f(x) = \tan^2 x$	$\left(\dfrac{\pi}{4}, 1\right)$

Writing **In Exercises 65–68, the graphs of a function f and its derivative f' are given. Label the graphs as f or f' and write a short paragraph stating the criteria used in making the selection.**

65.



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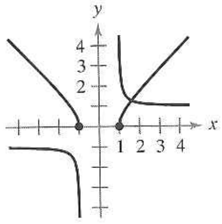

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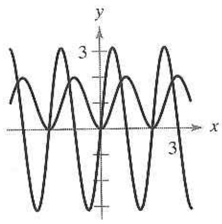

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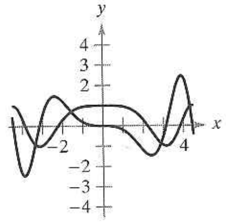

In Exercises 69–72, find the second derivative of the function.

69. $f(x) = 2(x^2 - 1)^3$

70. $f(x) = \dfrac{1}{x - 2}$

71. $f(x) = \sin x^2$

72. $f(x) = \sec^2 \pi x$

73. ***Think About It*** Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$, find $f'(5)$ (if possible) for each of the following. If it is not possible, state what additional information is required.

(a) $f(x) = g(x)h(x)$ (b) $f(x) = g(h(x))$

(c) $f(x) = \dfrac{g(x)}{h(x)}$ (d) $f(x) = [g(x)]^3$

74. (a) Find the derivative of the function $g(x) = \sin^2 x + \cos^2 x$ in two ways.

(b) For $f(x) = \sec^2 x$ and $g(x) = \tan^2 x$, show that $f'(x) = g'(x)$.

75. ***Doppler Effect*** The frequency F of a fire truck siren heard by a stationary observer is

$$F = \frac{132{,}400}{331 \pm v}$$

where $\pm v$ represents the velocity of the accelerating fire truck (see figure). Find the rate of change of F with respect to v when

(a) the fire truck is approaching at a velocity of 30 m/s (use $-v$).

(b) the fire truck is moving away at a velocity of 30 m/s (use $+v$).

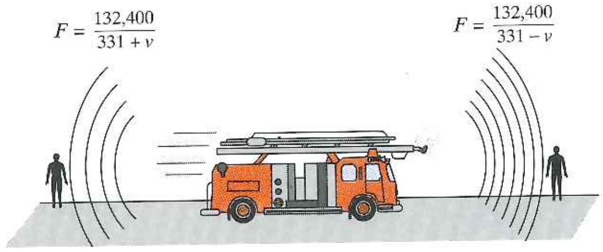

Figure for 75

76. ***Harmonic Motion*** The displacement from equilibrium of an object in harmonic motion on the end of a spring is

$$y = \tfrac{1}{3}\cos 12t - \tfrac{1}{4}\sin 12t$$

where y is measured in feet and t is the time in seconds. Determine the position and velocity of the object when $t = \pi/8$.

77. ***Pendulum*** A 15-centimeter pendulum moves according to the equation

$$\theta = 0.2\cos 8t$$

where θ is the angular displacement from the vertical in radians and t is the time in seconds. Determine the maximum angular displacement and the rate of change of θ when $t = 3$ seconds.

78. ***Wave Motion*** A buoy oscillates in simple harmonic motion

$$y = A\cos \omega t$$

as waves move past it. The buoy moves a total of 3.5 feet (vertically) from its low point to its high point. It returns to its high point every 10 seconds.

(a) Write an equation describing the motion of the buoy if it is at its high point at $t = 0$.

(b) Determine the velocity of the buoy as a function of t.

79. ***Circulatory System*** The speed S of blood that is r centimeters from the center of an artery is

$$S = C(R^2 - r^2)$$

where C is a constant, R is the radius of the artery, and S is measured in centimeters per second. Suppose a drug is administered and the artery begins to dilate at a rate of dR/dt. At a constant distance r, find the rate at which S changes with respect to t for $C = 1.76 \times 10^5$, $R = 1.2 \times 10^{-2}$, and $dR/dt = 10^{-5}$.

80. ***Modeling Data*** The normal daily maximum temperature T (in degrees Fahrenheit) for Denver, Colorado is given in the table. *(Source: National Oceanic and Atmosphere Administration)*

Month	Jan	Feb	Mar	Apr	May	Jun
Temperature	43.2	46.6	52.2	61.8	70.8	81.4

Month	Jul	Aug	Sep	Oct	Nov	Dec
Temperature	88.2	85.8	76.9	66.3	52.5	44.5

(a) Use a graphing utility to plot the data and find a model for the data of the form

$$T(t) = a + b \sin(\pi t/6 - c)$$

where T is the temperature and t is the time in months, with $t = 1$ corresponding to January.

(b) Use a graphing utility to graph the model. How well does the model fit the data?

(c) Find T' and use a graphing utility to graph the derivative.

(d) Based on the graph of the derivative, during what times does the temperature change most rapidly? Most slowly? Do your answers agree with your observations of temperature changes? Explain.

81. ***Think About It*** The table gives some values of the derivative of an unknown function f. Complete the table by finding (if possible) the derivative of each of the following transformations of f.

(a) $g(x) = f(x) - 2$ (b) $h(x) = 2f(x)$

(c) $r(x) = f(-3x)$ (d) $s(x) = f(x + 2)$

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$						
$h'(x)$						
$r'(x)$						
$s'(x)$						

82. ***Finding a Pattern*** Consider the function $f(x) = \sin \beta x$, where β is a constant.

(a) Find the first-, second-, third-, and fourth-order derivatives of the function.

(b) Verify that the function and its second derivative satisfy the equation $f''(x) + \beta^2 f(x) = 0$.

(c) Use the results in part (a) to write general rules for the even- and odd-order derivatives

$$f^{(2k)}(x) \text{ and } f^{(2k-1)}(x).$$

[*Hint:* $(-1)^k$ is positive if k is even and negative if k is odd.]

83. ***Conjecture*** Let f be a differentiable function of period p.

(a) Is the function f' periodic? Verify your answer.

(b) Consider the function $g(x) = f(2x)$. Is the function $g'(x)$ periodic? Verify your answer.

84. Show that the derivative of an odd function is even. That is, if $f(-x) = -f(x)$, then $f'(-x) = f'(x)$.

85. The geometric mean of x and $x + n$ is $g = \sqrt{x(x + n)}$, and the arithmetic mean is $a = [x + (x + n)]/2$. Show that

$$\frac{dg}{dx} = \frac{a}{g}.$$

86. Let u be a differentiable function of x. Use the fact that $|u| = \sqrt{u^2}$ to prove that

$$\frac{d}{dx}[|u|] = u'\frac{u}{|u|}, \quad u \neq 0.$$

In Exercises 87–90, use the result of Exercise 86 to find the derivative of the function.

87. $g(x) = |2x - 3|$ **88.** $f(x) = |x^2 - 4|$

89. $h(x) = |x| \cos x$ **90.** $f(x) = |\sin x|$

Linear and Quadratic Approximations **The linear and quadratic approximations of a function f at $x = a$ are**

$$P_1(x) = f'(a)(x - a) + f(a) \text{ and}$$

$$P_2(x) = \frac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a).$$

In Exercises 91 and 92, (a) find the specified linear and quadratic approximations of f, (b) use a graphing utility to graph f and the approximations, (c) determine whether P_1 or P_2 is the better approximation, and (d) state how the accuracy changes as you move farther from $x = a$.

91. $f(x) = \sin \dfrac{x}{2}$, $a = \pi$ **92.** $f(x) = \sec 2x$, $a = \dfrac{\pi}{6}$

True or False? **In Exercises 93–95, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

93. If $y = (1 - x)^{1/2}$, then $y' = \frac{1}{2}(1 - x)^{-1/2}$.

94. If $f(x) = \sin^2(2x)$, then $f'(x) = 2(\sin 2x)(\cos 2x)$.

95. If y is a differentiable function of u, u is a differentiable function of v, and v is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$