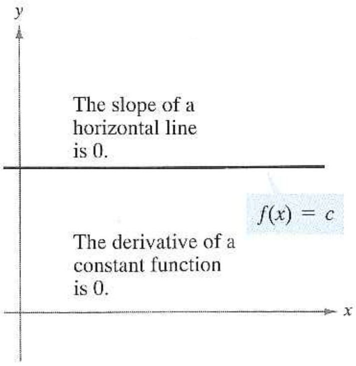
SECTION 2.2 Basic Differentiation Rules and Rates of Change

The Constant Rule • The Power Rule • The Constant Multiple Rule • The Sum and Difference Rules • Derivatives of Sine and Cosine Functions • Rates of Change

The Constant Rule



The Constant Rule
Figure 2.14

NOTE In Figure 2.14, note that the Constant Rule is equivalent to saying that the slope of a horizontal line is 0. This demonstrates the relationship between slope and derivative.

In Section 2.1 you used the limit definition to find derivatives. In this and the next two sections you will be introduced to several "differentiation rules" that allow you to find derivatives without the *direct* use of the limit definition.

THEOREM 2.2 The Constant Rule

The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0.$$

Proof Let $f(x) = c$. Then, by the limit definition of the derivative,

$$\begin{aligned}\frac{d}{dx}[c] &= f'(x)\\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x}\\ &= 0.\end{aligned}$$

EXAMPLE 1 Using the Constant Rule

Function	*Derivative*
a. $y = 7$	$\frac{dy}{dx} = 0$
b. $f(x) = 0$	$f'(x) = 0$
c. $s(t) = -3$	$s'(t) = 0$
d. $y = k\pi^2$, k is constant	$y' = 0$

EXPLORATION

Writing a Conjecture Use the definition of the derivative given in Section 2.1 to find the derivative of each of the following. What patterns do you see? Use your results to write a conjecture about the derivative of $f(x) = x^n$.

(a) $f(x) = x^1$ **(b)** $f(x) = x^2$ **(c)** $f(x) = x^3$

(d) $f(x) = x^4$ **(e)** $f(x) = x^{1/2}$ **(f)** $f(x) = x^{-1}$

The Power Rule

Before proving the next rule, we review the procedure for expanding a binomial.

$$(x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2$$
$$(x + \Delta x)^3 = x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

The general binomial expansion for a positive integer n is

$$(x + \Delta x)^n = x^n + nx^{n-1}(\Delta x) + \underbrace{\frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n}_{(\Delta x)^2 \text{ is a factor of these terms.}}.$$

This binomial expansion is used in proving a special case of the Power Rule.

THEOREM 2.3 The Power Rule

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing 0.

Proof If n is a positive integer greater than 1, then the binomial expansion produces the following.

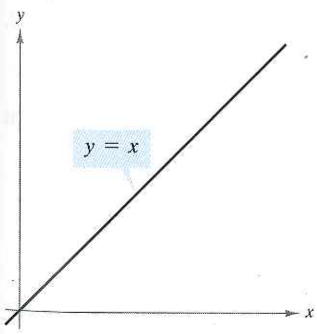
$$\frac{d}{dx}[x^n] = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}(\Delta x) + \frac{n(n-1)x^{n-2}}{2}(\Delta x)^2 + \cdots + (\Delta x)^n - x^n}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left[nx^{n-1} + \frac{n(n-1)x^{n-2}}{2}(\Delta x) + \cdots + (\Delta x)^{n-1} \right]$$
$$= nx^{n-1} + 0 + \cdots + 0$$
$$= nx^{n-1}$$

This proves the case for which n is a positive integer greater than 1. We leave it to you to prove the case for $n = 1$. Example 7 in Section 2.3 proves the case for which n is a negative integer. Exercise 59 in Section 2.5 proves the case for which n is rational. (In Section 5.5, the Power Rule will be extended to cover irrational values of n.)

In the Power Rule, the case for which $n = 1$ is best thought of as a separate differentiation rule. That is,

$$\frac{d}{dx}[x] = 1. \qquad \text{Power Rule when } n = 1$$

This rule is consistent with the fact that the slope of the line $y = x$ is 1, as shown in Figure 2.15.



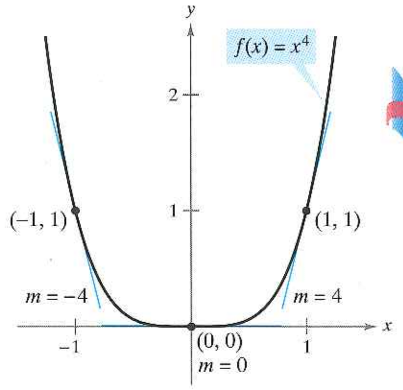
The slope of the line $y = x$ is 1.
Figure 2.15

EXAMPLE 2 Using the Power Rule

Function	*Derivative*
a. $f(x) = x^3$	$f'(x) = 3x^2$
b. $g(x) = \sqrt[3]{x}$	$g'(x) = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$
c. $y = \frac{1}{x^2}$	$\frac{dy}{dx} = \frac{d}{dx}[x^{-2}] = (-2)x^{-3} = -\frac{2}{x^3}$

In Example 2c, note that *before* differentiating, $1/x^2$ was rewritten as x^{-2}. Rewriting is the first step in *many* differentiation problems.

Given: $y = \frac{1}{x^2}$ ⇨ Rewrite: $y = x^{-2}$ ⇨ Differentiate: $\frac{dy}{dx} = (-2)x^{-3}$ ⇨ Simplify: $\frac{dy}{dx} = -\frac{2}{x^3}$



The slope of a graph at a point is the value of the derivative at that point.
Figure 2.16

EXAMPLE 3 Finding the Slope of a Graph

Find the slope of the graph of $f(x) = x^4$ when

a. $x = -1$ **b.** $x = 0$ **c.** $x = 1$.

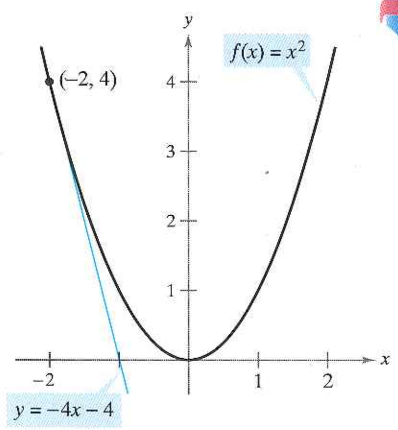
Solution The derivative of f is $f'(x) = 4x^3$.

a. When $x = -1$, the slope is $f'(-1) = 4(-1)^3 = -4$.

b. When $x = 0$, the slope is $f'(0) = 4(0)^3 = 0$.

c. When $x = 1$, the slope is $f'(1) = 4(1)^3 = 4$.

In Figure 2.16, note that the slope of the graph is negative at the point $(-1, 1)$, the slope is zero at the point $(0, 0)$, and the slope is positive at the point $(1, 1)$.



The line $y = -4x - 4$ is tangent to the graph of $f(x) = x^2$ at the point $(-2, 4)$.
Figure 2.17

EXAMPLE 4 Finding an Equation of a Tangent Line

Find an equation of the tangent line to the graph of $f(x) = x^2$ when $x = -2$.

Solution To find the *point* on the graph of f, evaluate the original function at $x = -2$.

$$(-2, f(-2)) = (-2, 4) \qquad \text{Point on graph}$$

To find the *slope* of the graph when $x = -2$, evaluate the derivative, $f'(x) = 2x$, at $x = -2$.

$$m = f'(-2) = -4 \qquad \text{Slope of graph at } (-2, 4)$$

Now, using the point-slope form of the equation of a line, you can write

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$
$$y - 4 = -4[x - (-2)] \qquad \text{Substitute for } y_1, m, \text{ and } x_1.$$
$$y = -4x - 4. \qquad \text{Simplify.}$$

(See Figure 2.17.)

The Constant Multiple Rule

THEOREM 2.4 The Constant Multiple Rule

If f is a differentiable function and c is a real number, then cf is also differentiable and

$$\frac{d}{dx}[cf(x)] = cf'(x).$$

Proof

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{\Delta x \to 0} \frac{cf(x + \Delta x) - cf(x)}{\Delta x}\\ &= \lim_{\Delta x \to 0} c\left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right]\\ &= c\left[\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}\right]\\ &= cf'(x)\end{aligned}$$

Informally, the Constant Multiple Rule states that constants can be factored out of the differentiation process, even if the constants appear in the denominator.

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)] = cf'(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{c}\right] = \frac{d}{dx}\left[\left(\frac{1}{c}\right)f(x)\right] = \left(\frac{1}{c}\right)\frac{d}{dx}[f(x)] = \left(\frac{1}{c}\right)f'(x)$$

EXAMPLE 5 Using the Constant Multiple Rule

Function	*Derivative*
a. $y = \dfrac{2}{x}$	$\dfrac{dy}{dx} = \dfrac{d}{dx}[2x^{-1}] = 2\dfrac{d}{dx}[x^{-1}] = 2(-1)x^{-2} = -\dfrac{2}{x^2}$
b. $f(t) = \dfrac{4t^2}{5}$	$f'(t) = \dfrac{d}{dt}\left[\dfrac{4}{5}t^2\right] = \dfrac{4}{5}\dfrac{d}{dt}[t^2] = \dfrac{4}{5}(2t) = \dfrac{8}{5}t$
c. $y = 2\sqrt{x}$	$\dfrac{dy}{dx} = \dfrac{d}{dx}[2x^{1/2}] = 2\left(\dfrac{1}{2}x^{-1/2}\right) = x^{-1/2} = \dfrac{1}{\sqrt{x}}$
d. $y = \dfrac{1}{2\sqrt[3]{x^2}}$	$\dfrac{dy}{dx} = \dfrac{d}{dx}\left[\dfrac{1}{2}x^{-2/3}\right] = \dfrac{1}{2}\left(-\dfrac{2}{3}\right)x^{-5/3} = -\dfrac{1}{3x^{5/3}}$
e. $y = -\dfrac{3x}{2}$	$y' = \dfrac{d}{dx}\left[-\dfrac{3}{2}x\right] = -\dfrac{3}{2}(1) = -\dfrac{3}{2}$

NOTE The Constant Multiple Rule and the Power Rule can be combined into one rule. The combination rule is $D_x[cx^n] = cnx^{n-1}$.

EXAMPLE 6 Using Parentheses When Differentiating

	Original Function	*Rewrite*	*Differentiate*	*Simplify*
a.	$y = \dfrac{5}{2x^3}$	$y = \dfrac{5}{2}(x^{-3})$	$y' = \dfrac{5}{2}(-3x^{-4})$	$y' = -\dfrac{15}{2x^4}$
b.	$y = \dfrac{5}{(2x)^3}$	$y = \dfrac{5}{8}(x^{-3})$	$y' = \dfrac{5}{8}(-3x^{-4})$	$y' = -\dfrac{15}{8x^4}$
c.	$y = \dfrac{7}{3x^{-2}}$	$y = \dfrac{7}{3}(x^2)$	$y' = \dfrac{7}{3}(2x)$	$y' = \dfrac{14x}{3}$
d.	$y = \dfrac{7}{(3x)^{-2}}$	$y = 63(x^2)$	$y' = 63(2x)$	$y' = 126x$

The Sum and Difference Rules

THEOREM 2.5 The Sum and Difference Rules

The derivative of the sum (or difference) of two differentiable functions is differentiable and is the sum (or difference) of their derivatives.

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \qquad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \qquad \text{Difference Rule}$$

Proof A proof of the Sum Rule follows from Theorem 1.2. (The Difference Rule can be proved in a similar way.)

$$\begin{aligned}\frac{d}{dx}[f(x) + g(x)] &= \lim_{\Delta x \to 0} \frac{[f(x + \Delta x) + g(x + \Delta x)] - [f(x) + g(x)]}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) + g(x + \Delta x) - f(x) - g(x)}{\Delta x}\\ &= \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x}\right]\\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}\\ &= f'(x) + g'(x)\end{aligned}$$

The Sum and Difference Rules can be extended to any finite number of functions. For instance, if $F(x) = f(x) + g(x) - h(x) - k(x)$, then $F'(x) = f'(x) + g'(x) - h'(x) - k'(x)$.

EXAMPLE 7 Using the Sum and Difference Rules

	Function	*Derivative*
a.	$f(x) = x^3 - 4x + 5$	$f'(x) = 3x^2 - 4$
b.	$g(x) = -\dfrac{x^4}{2} + 3x^3 - 2x$	$g'(x) = -2x^3 + 9x^2 - 2$

FOR FURTHER INFORMATION For the outline of a geometric proof of the derivatives of the sine and cosine functions, see the article "The Spider's Spacewalk Derivation of sin′ and cos′" by Tim Hesterberg in the March 1995 issue of *The College Mathematics Journal.*

Derivatives of Sine and Cosine Functions

In Section 1.3, you studied the following limits.

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1 \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0$$

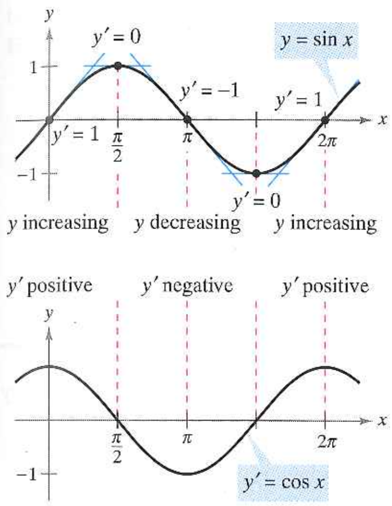
These two limits can be used to prove differentiation rules for the sine and cosine functions. (The derivatives of the other four trigonometric functions are discussed in Section 2.3.)

THEOREM 2.6 Derivatives of Sine and Cosine Functions

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

Proof

$$\begin{aligned}
\frac{d}{dx}[\sin x] &= \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cos x \sin \Delta x - (\sin x)(1 - \cos \Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[(\cos x)\left(\frac{\sin \Delta x}{\Delta x}\right) - (\sin x)\left(\frac{1 - \cos \Delta x}{\Delta x}\right) \right] \\
&= \cos x \left(\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} \right) - \sin x \left(\lim_{\Delta x \to 0} \frac{1 - \cos \Delta x}{\Delta x} \right) \\
&= (\cos x)(1) - (\sin x)(0) \\
&= \cos x
\end{aligned}$$

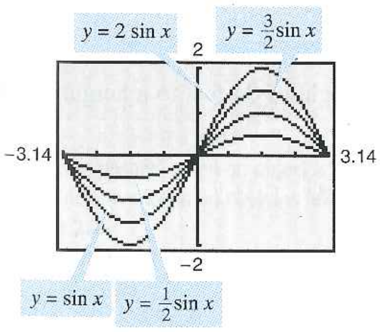


The derivative of the sine function is the cosine function.
Figure 2.18

This differentiation rule is shown graphically in Figure 2.18. Note that for each x, the *slope* of the sine curve is equal to the *value* of the cosine. The proof of the second rule is left as an exercise (see Exercise 93).

EXAMPLE 8 Derivatives Involving Sines and Cosines

Function	*Derivative*
a. $y = 2 \sin x$	$y' = 2 \cos x$
b. $y = \dfrac{\sin x}{2} = \dfrac{1}{2} \sin x$	$y' = \dfrac{1}{2} \cos x = \dfrac{\cos x}{2}$
c. $y = x + \cos x$	$y' = 1 - \sin x$



$\frac{d}{dx}[a \sin x] = a \cos x$
Figure 2.19

TECHNOLOGY A graphing utility can provide insight into the interpretation of a derivative. For instance, Figure 2.19 shows the graphs of

$$y = a \sin x$$

for $a = \frac{1}{2}, 1, \frac{3}{2}$, and 2. Estimate the slope of each graph at the point $(0, 0)$. Then verify your estimates analytically by evaluating the derivative of each function when $x = 0$.

Rates of Change

You have seen how the derivative is used to determine slope. The derivative can also be used to determine the rate of change of one variable with respect to another. Applications involving rates of change occur in a wide variety of fields. A few examples are population growth rates, production rates, water flow rates, velocity, and acceleration.

A common use of rate of change is to describe the motion of an object moving in a straight line. In such problems, it is customary to use either a horizontal or a vertical line with a designated origin to represent the line of motion. On such lines, movement to the right (or upward) is considered to be in the positive direction, and movement to the left (or downward) is considered to be in the negative direction.

The function s that gives the position (relative to the origin) of an object as a function of time t is called a **position function.** If, over a period of time Δt, the object changes its position by the amount $\Delta s = s(t + \Delta t) - s(t)$, then, by the familiar formula

$$\text{Rate} = \frac{\text{distance}}{\text{time}}$$

the **average velocity** is

$$\frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta s}{\Delta t}. \qquad \text{Average velocity}$$

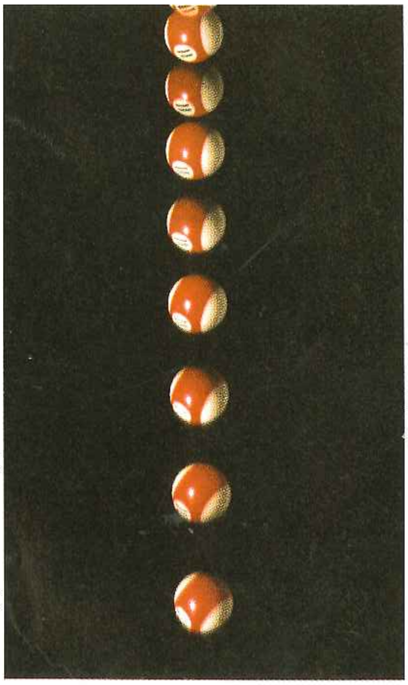
EXAMPLE 9 Finding Average Velocity of a Falling Object

If a billiard ball is dropped from a height of 100 feet, its height s at time t is given by the position function

$$s = -16t^2 + 100 \qquad \text{Position function}$$

where s is measured in feet and t is measured in seconds. Find the average velocity over each of the following time intervals.

a. $[1, 2]$ **b.** $[1, 1.5]$ **c.** $[1, 1.1]$



Richard Megna/Fundamental Photographs

Time-lapse photograph of a free-falling billiard ball.

Solution

a. For the interval $[1, 2]$ the object falls from a height of $s(1) = -16(1)^2 + 100 = 84$ feet to a height of $s(2) = -16(2)^2 + 100 = 36$ feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{36 - 84}{2 - 1} = \frac{-48}{1} = -48 \text{ feet per second.}$$

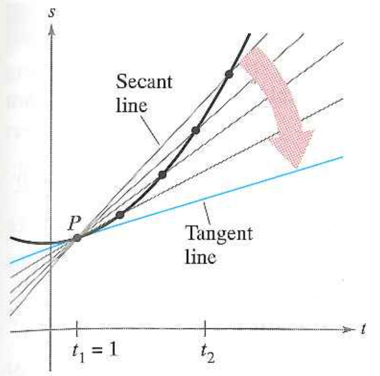
b. For the interval $[1, 1.5]$, the object falls from a height of 84 feet to a height of 64 feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{64 - 84}{1.5 - 1} = \frac{-20}{0.5} = -40 \text{ feet per second.}$$

c. For the interval $[1, 1.1]$, the object falls from a height of 84 feet to a height of 80.64 feet. The average velocity is

$$\frac{\Delta s}{\Delta t} = \frac{80.64 - 84}{1.1 - 1} = \frac{-3.36}{0.1} = -33.6 \text{ feet per second.}$$

Note that the average velocities are *negative*, indicating that the object is moving downward.



The average velocity between t_1 and t_2 is the slope of the secant line, and the instantaneous velocity at t_1 is the slope of the tangent line.
Figure 2.20

Suppose that in Example 9 you wanted to find the *instantaneous* velocity (or simply the velocity) of the object when $t = 1$. Just as you can approximate the slope of the tangent line by calculating the slope of the secant line, you can approximate the velocity at $t = 1$ by calculating the average velocity over a small interval $[1, 1 + \Delta t]$ (see Figure 2.20). By taking the limit as Δt approaches zero, you obtain the velocity when $t = 1$. Try doing this—you will find that the velocity when $t = 1$ is -32 feet per second.

In general, if $s = s(t)$ is the position function for an object moving along a straight line, the **velocity** of the object at time t is

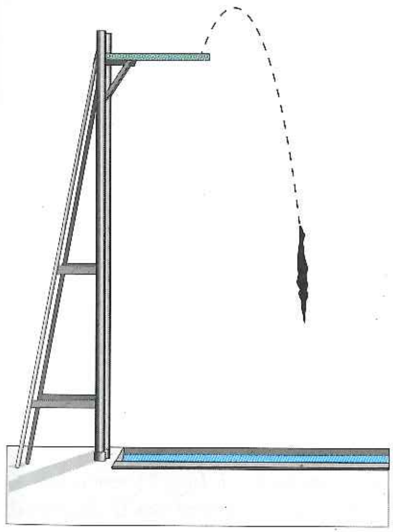
$$v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t).$$ Velocity function

In other words, the velocity function is the derivative of the position function. (Velocity can be negative, zero, or positive. The **speed** of an object is the absolute value of its velocity. Speed cannot be negative.)

The position of a free-falling object (neglecting air resistance) under the influence of gravity can be represented by the equation

$$s(t) = \frac{1}{2}gt^2 + v_0t + s_0$$ Position function

where s_0 is the initial height of the object, v_0 is the initial velocity of the object, and g is the acceleration due to gravity. On earth, the value of g is approximately -32 feet per second per second or -9.8 meters per second per second.



Velocity is positive when an object is rising, and is negative when an object is falling.
Figure 2.21

NOTE In Figure 2.21, note that the diver moves upward for the first half-second because the velocity is positive for $0 < t < \frac{1}{2}$. When the velocity is 0, the diver has reached the maximum height of the dive.

EXAMPLE 10 Using the Derivative to Find Velocity

At time $t = 0$, a diver jumps from a diving board that is 32 feet above the water (see Figure 2.21). The position of the diver is given by

$$s(t) = -16t^2 + 16t + 32$$

where s is measured in feet and t is measured in seconds.

a. When does the diver hit the water?

b. What is the diver's velocity at impact?

Solution

a. To find when the diver hits the water, let $s = 0$ and solve for t.

$$-16t^2 + 16t + 32 = 0$$
$$-16(t + 1)(t - 2) = 0$$
$$t = -1 \text{ or } 2$$

Because $t \geq 0$, choose the positive value to conclude that the diver hits the water at $t = 2$ seconds.

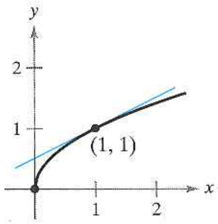
b. The velocity at time t is given by the derivative $s'(t) = -32t + 16$. Therefore, the velocity at time $t = 2$ is

$$s'(2) = -32(2) + 16 = -48 \text{ feet per second.}$$

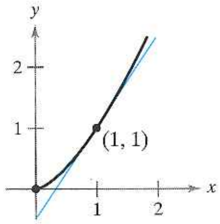
EXERCISES FOR SECTION 2.2

In Exercises 1 and 2, find the slope of the tangent line to $y = x^n$ at the point (1, 1).

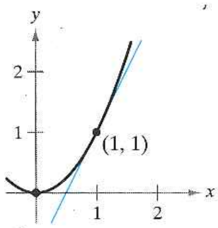
1. (a) $y = x^{1/2}$



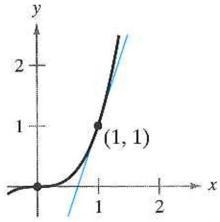
(b) $y = x^{3/2}$



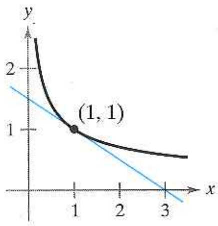
(c) $y = x^2$



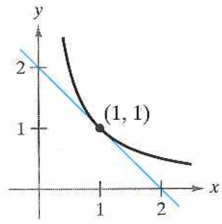
(d) $y = x^3$



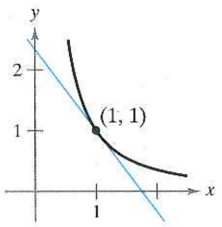
2. (a) $y = x^{-1/2}$



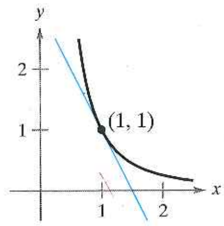
(b) $y = x^{-1}$



(c) $y = x^{-3/2}$



(d) $y = x^{-2}$



In Exercises 3–16, find the derivative of the function.

3. $y = 3$
4. $f(x) = -2$
5. $f(x) = x + 1$
6. $g(x) = 3x - 1$
7. $g(x) = x^2 + 4$
8. $y = t^2 + 2t - 3$
9. $f(t) = -2t^2 + 3t - 6$
10. $y = x^3 - 9$
11. $s(t) = t^3 - 2t + 4$
12. $f(x) = 2x^3 - x^2 + 3x$
13. $y = x^2 - \frac{1}{2}\cos x$
14. $y = 5 + \sin x$
15. $y = \dfrac{1}{x} - 3\sin x$
16. $g(t) = \pi \cos t$

In Exercises 17–22, complete the table, using Example 6 as a model.

	Original Function	*Rewrite*	*Differentiate*	*Simplify*
17.	$y = \dfrac{1}{3x^3}$			
18.	$y = \dfrac{2}{3x^2}$			
19.	$y = \dfrac{1}{(3x)^3}$			
20.	$y = \dfrac{\pi}{(3x)^2}$			
21.	$y = \dfrac{\sqrt{x}}{x}$			
22.	$y = \dfrac{4}{x^{-3}}$			

In Exercises 23–30, find the value of the derivative of the function at the indicated point. Use the derivative feature of a graphing utility to confirm your results.

	Function	*Point*
23.	$f(x) = \dfrac{1}{x}$	$(1, 1)$
24.	$f(t) = 3 - \dfrac{3}{5t}$	$(\frac{3}{5}, 2)$
25.	$f(x) = -\frac{1}{2} + \frac{7}{5}x^3$	$(0, -\frac{1}{2})$
26.	$y = 3x\left(x^2 - \dfrac{2}{x}\right)$	$(2, 18)$
27.	$y = (2x + 1)^2$	$(0, 1)$
28.	$f(x) = 3(5 - x)^2$	$(5, 0)$
29.	$f(\theta) = 4\sin\theta - \theta$	$(0, 0)$
30.	$g(t) = 2 + 3\cos t$	$(\pi, -1)$

In Exercises 31–42, find the derivative of the function.

31. $f(x) = x^3 - 3x - 2x^{-4}$
32. $f(x) = x^2 - 3x - 3x^{-2}$
33. $g(t) = t^2 - \dfrac{4}{t}$
34. $f(x) = x + \dfrac{1}{x^2}$
35. $f(x) = \dfrac{x^3 - 3x^2 + 4}{x^2}$
36. $h(x) = \dfrac{2x^2 - 3x + 1}{x}$
37. $y = x(x^2 + 1)$
38. $f(x) = \sqrt[3]{x} + \sqrt[5]{x}$
39. $h(s) = s^{4/5}$
40. $f(t) = t^{1/3} - 1$
41. $f(x) = 4\sqrt{x} + 3\cos x$
42. $f(x) = 2\sin x + 3\cos x$

In Exercises 43–46, (a) find an equation of the tangent line to the graph of f at the indicated point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the derivative feature of a graphing utility to confirm your results.

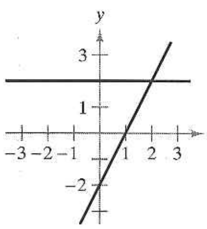
Function	*Point*
43. $y = x^4 - 3x^2 + 2$	$(1, 0)$
44. $y = x^3 + x$	$(-1, -2)$
45. $f(x) = \dfrac{1}{\sqrt[3]{x^2}}$	$\left(8, \frac{1}{4}\right)$
46. $y = (x^2 + 2x)(x + 1)$	$(1, 6)$

In Exercises 47–52, determine the point(s) (if any) at which the function has a horizontal tangent line.

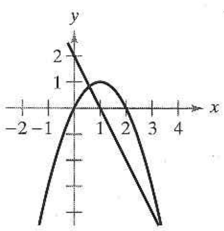
47. $y = x^4 - 8x^2 + 2$
48. $y = x^3 + x$
49. $y = \dfrac{1}{x^2}$
50. $y = x^2 + 1$
51. $y = x + \sin x, \quad 0 \le x < 2\pi$
52. $y = \sqrt{3}x + 2\cos x, \quad 0 \le x < 2\pi$

Writing **In Exercises 53 and 54, the graphs of a function f and its derivative f' are given on the same set of coordinate axes. Label the graphs as f or f' and write a short paragraph stating the criteria used in making the selection.**

53.



54.



55. Sketch the graphs of the two equations $y = x^2$ and $y = -x^2 + 6x - 5$, and sketch the two lines that are tangent to both graphs. Find the equations of these lines.

56. Show that the graphs of the two equations $y = x$ and $y = 1/x$ have tangent lines that are perpendicular to each other at their point of intersection.

In Exercises 57 and 58, find an equation of the tangent line to the graph of the function f through the point (x_0, y_0) not on the graph. To find the point of tangency (x, y) on the graph of f, solve the equation

$$f'(x) = \frac{y_0 - y}{x_0 - x}.$$

57. $f(x) = \sqrt{x}$

$(x_0, y_0) = (-4, 0)$

58. $f(x) = \dfrac{2}{x}$

$(x_0, y_0) = (5, 0)$

59. ***Linear Approximation*** Consider the function $f(x) = x^{3/2}$ with the solution point $(4, 8)$.

(a) Use a graphing utility to obtain the graph of f. Use the zoom feature to obtain successive magnifications of the graph in the neighborhood of the point $(4, 8)$. After zooming in a few times, the graph should appear nearly linear. Use the trace feature to determine the coordinates of a point "near" $(4, 8)$. Find an equation of the secant line $S(x)$ through the two points.

(b) Find the equation of the line

$$T(x) = f'(4)(x - 4) + f(4)$$

tangent to the graph of f passing through the given point. Why are the linear functions S and T nearly the same?

(c) Use a graphing utility to graph f and T on the same set of coordinate axes. Note that T is a "good" approximation of f when x is "close to" 4. What happens to the accuracy of the approximation as you move farther away from the point of tangency?

(d) Demonstrate the conclusion in part (c) by completing the table.

Δx	-3	-2	-1	-0.5	-0.1	0
$f(x)$						
$T(x)$						

Δx	0.1	0.5	1	2	3
$f(x)$					
$T(x)$					

60. ***Linear Approximation*** Repeat Exercise 59 for the function $f(x) = x^3$ where $T(x)$ is the line tangent to the graph at the point $(1, 1)$. Explain why the accuracy of the linear approximation decreases more rapidly than in Exercise 59.

True or False? **In Exercises 61–64, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

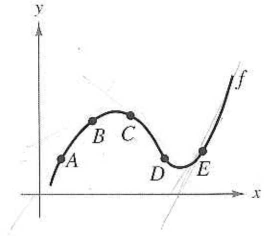
61. If $f'(x) = g'(x)$, then $f(x) = g(x)$.
62. If $f(x) = g(x) + c$, then $f'(x) = g'(x)$.
63. If $y = \pi^2$, then $dy/dx = 2\pi$.
64. If $y = x/\pi$, then $dy/dx = 1/\pi$.

In Exercises 65–68, find the average rate of change of the function over the indicated interval. Compare this average rate of change with the instantaneous rates of change at the endpoints of the interval.

Function	*Interval*
65. $f(t) = 2t + 7$	$[1, 2]$
66. $f(t) = t^2 - 3$	$[2, 2.1]$

Function	*Interval*
67. $f(x) = \dfrac{-1}{x}$	$[1, 2]$
68. $f(x) = \sin x$	$\left[0, \dfrac{\pi}{6}\right]$

69. ***Think About It*** Use the graph of f to answer each question.



(a) Between which two consecutive points is the average rate of change of the function greatest?

(b) Is the average rate of change of the function between A and B greater than or less than the instantaneous rate of change at B?

(c) Sketch a tangent line to the graph between the points C and D such that the slope of the tangent line is the same as the average rate of change of the function between C and D.

(d) Give any sets of consecutive points for which the average rates of change of the function are approximately equal.

70. ***Think About It*** Sketch the graph of a function f such that $f' > 0$ for all x and the rate of change of the function is decreasing.

Vertical Motion **In Exercises 71 and 72, use the position function $s(t) = -16t^2 + v_0 t + s_0$ for free-falling objects.**

71. A silver dollar is dropped from the top of a building, which is 1362 feet tall.

(a) Determine the position and velocity functions for the coin.

(b) Determine the average velocity on the interval $[1, 2]$.

(c) Find the instantaneous velocities when $t = 1$ and $t = 2$.

(d) Find the time required for the coin to reach ground level.

(e) Find the velocity of the dollar just before it hits the ground.

72. A ball is thrown straight down from the top of a 220-foot building with an initial velocity of -22 feet per second. What is its velocity after 3 seconds? What is its velocity after falling 108 feet?

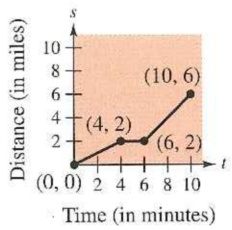
Vertical Motion **In Exercises 73 and 74, use the position function $s(t) = -4.9t^2 + v_0 t + s_0$ for free-falling objects.**

73. A projectile is shot upward from the surface of the earth with an initial velocity of 120 meters per second. What is its velocity after 5 seconds? After 10 seconds?

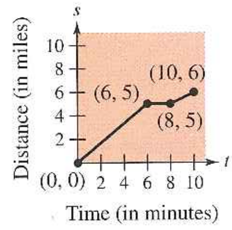
74. To estimate the height of a building, a stone is dropped from the top of the building into a pool of water at ground level. How high is the building if the splash is seen 6.8 seconds after the stone is dropped?

Think About It **In Exercises 75 and 76, the graph of a position function is given. It represents the distance in miles that a person drives during a 10-minute trip to work. Make a sketch of the corresponding velocity function.**

75.

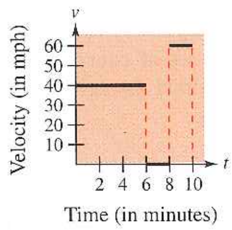


76.

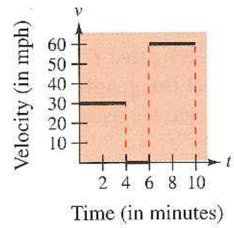


Think About It **In Exercises 77 and 78, the graph of a velocity function is given. It represents the velocity in miles per hour during a 10-minute drive to work. Make a sketch of the corresponding position function.**

77.



78.



79. ***Modeling Data*** The stopping distance of an automobile traveling at a speed v (kilometers per hour) is the distance R (meters) the car travels during the reaction time of the driver plus the distance B (meters) the car travels after the brakes are applied. The table shows the results of an experiment.

v	20	40	60	80	100
R	3.3	6.7	10.0	13.3	16.7
B	2.3	8.9	20.2	35.9	56.7

(a) Use the regression capabilities of a graphing utility to find a linear model for reaction time.

(b) Use the regression capabilities of a graphing utility to find a quadratic model for braking time.

(c) Determine the polynomial giving the total stopping distance T.

(d) Use a graphing utility to graph the functions R, B, and T in the same viewing rectangle.

(e) Find the derivative of T and the rate of change of the total stopping distance for $v = 40$, $v = 80$, and $v = 100$.

(f) Use the results of this exercise to draw conclusions about the total stopping distance as speed increases.

80. ***Velocity*** Verify that the average velocity over the time interval $[t_0 - \Delta t, t_0 + \Delta t]$ is the same as the instantaneous velocity at $t = t_0$ for the position function.

$$s(t) = -\tfrac{1}{2}at^2 + c.$$

81. ***Area*** The area of a square with sides of length s is given by $A = s^2$. Find the rate of change of the area with respect to s when $s = 4$ meters.

82. ***Volume*** The volume of a cube with sides of length s is given by $V = s^3$. Find the rate of change of the volume with respect to s when $s = 4$ centimeters.

83. ***Inventory Management*** The annual inventory cost C for a certain manufacturer is

$$C = \frac{1{,}008{,}000}{Q} + 6.3Q$$

where Q is the order size when the inventory is replenished. Find the change in annual cost when Q is increased from 350 to 351, and compare this with the instantaneous rate of change when $Q = 350$.

84. ***Fuel Cost*** A car is driven 15,000 miles a year and gets x miles per gallon. Assume that the average fuel cost is \$1.25 per gallon. Find the annual cost of fuel C as a function of x and use this function to complete the table.

x	10	15	20	25	30	35	40
C							
$\frac{dC}{dx}$							

Who would benefit more from a 1-mile-per-gallon increase in fuel efficiency—the driver of a car that gets 15 miles per gallon or the driver of a car that gets 35 miles per gallon? Explain.

85. ***Writing*** The photo at the bottom of the page was taken at game 4 of the 1996 World Series between the New York Yankees and the Atlanta Braves. Do you think the runner was safe or out? Write a detailed explanation of your reasoning.

86. ***Newton's Law of Cooling*** This law states that the rate of change of the temperature of an object is proportional to the difference between the object's temperature T and the temperature T_a of the surrounding medium. Write an equation for this law.

87. Find an equation of the parabola $y = ax^2 + bx + c$ that passes through $(0, 1)$ and is tangent to the line $y = x - 1$ at $(1, 0)$.

88. Let (a, b) be an arbitrary point on the graph of $y = 1/x, x > 0$. Prove that the area of the triangle formed by the tangent line through (a, b) and the coordinate axes is 2.

89. Find the tangent line(s) to the curve $y = x^3 - 9x$ through the point $(1, -9)$.

90. Find the equation(s) of the tangent line(s) to the parabola $y = x^2$ through the given point.

(a) $(0, a)$ (b) $(a, 0)$

Are there any restrictions on the constant a?

91. Find a and b such that

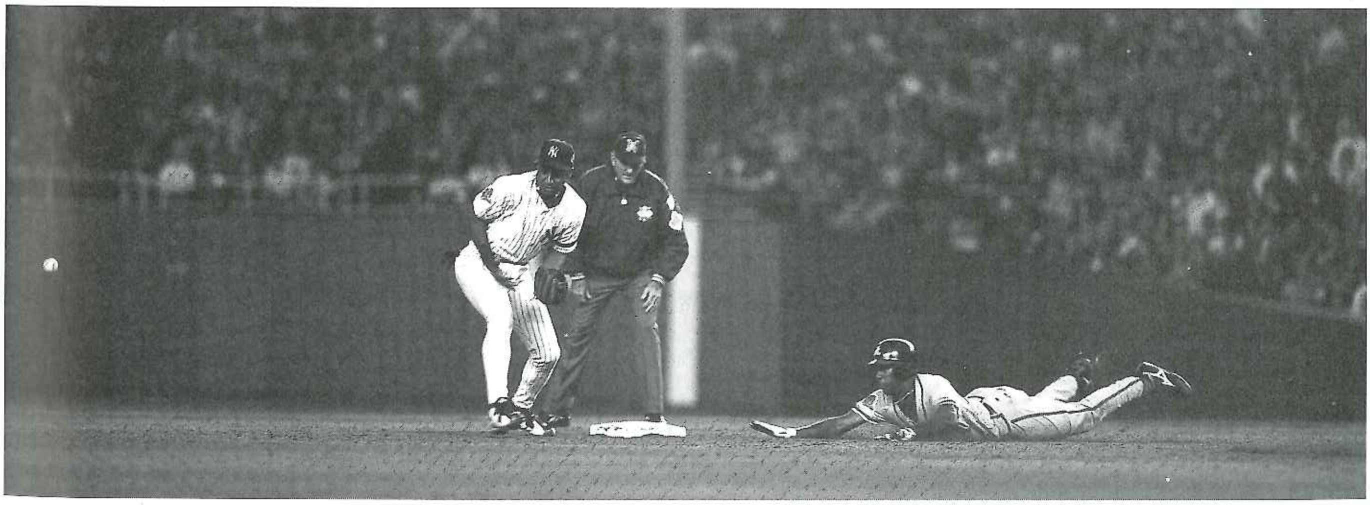
$$f(x) = \begin{cases} ax^3, & x \le 2 \\ x^2 + b, & x > 2 \end{cases}$$

is differentiable everywhere.

92. Where are the functions $f_1(x) = |\sin x|$ and $f_2(x) = \sin |x|$ differentiable?

93. Prove that $\frac{d}{dx}[\cos x] = -\sin x$.

FOR FURTHER INFORMATION For a geometric interpretation of the derivatives of trigonometric functions, see the article "Sines and Cosines of the Times" by Victor J. Katz in the April 1995 issue of *Math Horizons*.



David Burnett/Contact Press Images, Inc.