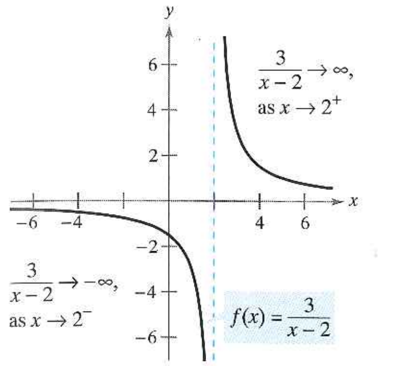
SECTION 1.5 Infinite Limits

Infinite Limits • Vertical Asymptotes

Infinite Limits

Let f be the function given by

$$f(x) = \frac{3}{x-2}.$$



$f(x)$ increases and decreases without bound as x approaches 2.
Figure 1.37

From Figure 1.37 and the table, you can see that $f(x)$ *decreases without bound* as x approaches 2 from the left, and $f(x)$ *increases without bound* as x approaches 2 from the right. This behavior is denoted as

$$\lim_{x\to 2^-} \frac{3}{x-2} = -\infty \qquad f(x) \text{ decreases without bound as } x \text{ approaches 2 from the left.}$$

and

$$\lim_{x\to 2^+} \frac{3}{x-2} = \infty \qquad f(x) \text{ increases without bound as } x \text{ approaches 2 from the right.}$$

x approaches 2 from the left. → ← x approaches 2 from the right.

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
$f(x)$	−6	−30	−300	−3000	?	3000	300	30	6

$f(x)$ decreases without bound. → ← $f(x)$ increases without bound.

A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an **infinite limit.**

Definition of Infinite Limits

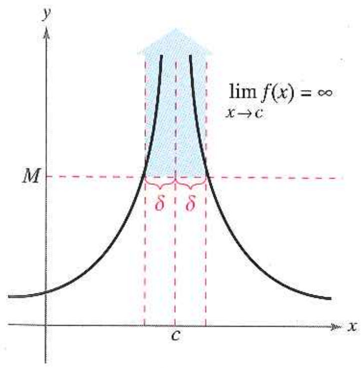
Let f be a function that is defined at every real number in some open interval containing c (except possibly at c itself). The statement

$$\lim_{x\to c} f(x) = \infty$$

means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$ (see Figure 1.38). Similarly, the statement

$$\lim_{x\to c} f(x) = -\infty$$

means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever $0 < |x - c| < \delta$. To define the **infinite limit from the left,** replace $0 < |x - c| < \delta$ by $c - \delta < x < c$. To define the **infinite limit from the right,** replace $0 < |x - c| < \delta$ by $c < x < c + \delta$.



Infinite limits
Figure 1.38

Be sure you see that the equal sign in the statement $\lim f(x) = \infty$ does not mean that the limit exists! On the contrary, it tells you how the limit *fails to exist* by denoting the unbounded behavior of $f(x)$ as x approaches c.

EXPLORATION

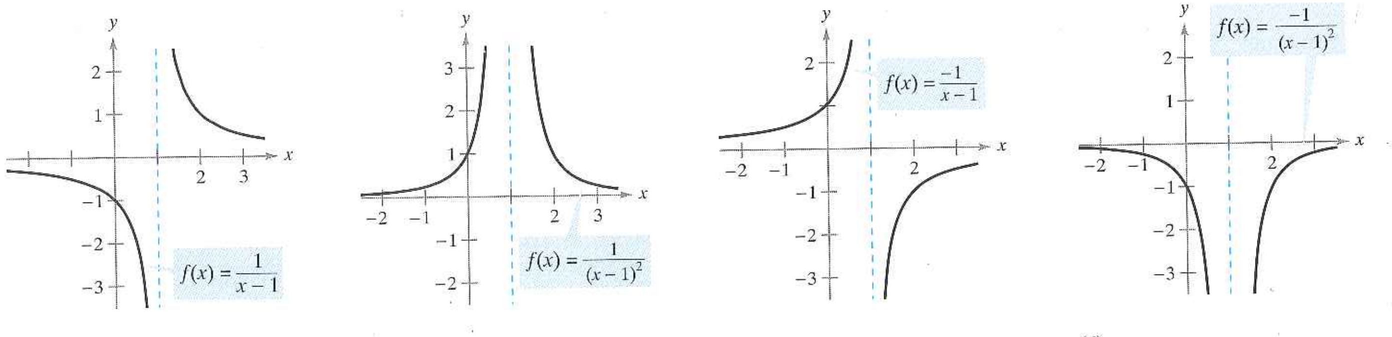
Use a graphing utility to graph each function. For each function, analytically find the single real number c that is not in the domain. Then graphically find the limit of $f(x)$ as x approaches c from the left and from the right.

(a) $f(x) = \dfrac{3}{x - 4}$ **(b)** $f(x) = \dfrac{1}{2 - x}$

(c) $f(x) = \dfrac{2}{(x - 3)^2}$ **(d)** $f(x) = \dfrac{-3}{(x + 2)^2}$

EXAMPLE 1 Determining Infinite Limits from a Graph

Use Figure 1.39 to determine the limit of each function as x approaches 1 from the left and from the right.



(a) (b) (c) (d)

Each graph has an asymptote at $x = 1$.
Figure 1.39

Solution

a. $\lim\limits_{x\to 1^-} \dfrac{1}{x - 1} = -\infty$ and $\lim\limits_{x\to 1^+} \dfrac{1}{x - 1} = \infty$

b. $\lim\limits_{x\to 1} \dfrac{1}{(x - 1)^2} = \infty$ Limit from each side is ∞.

c. $\lim\limits_{x\to 1^-} \dfrac{-1}{x - 1} = \infty$ and $\lim\limits_{x\to 1^+} \dfrac{-1}{x - 1} = -\infty$

d. $\lim\limits_{x\to 1} \dfrac{-1}{(x - 1)^2} = -\infty$ Limit from each side is $-\infty$.

Vertical Asymptotes

If it were possible to extend the graphs in Figure 1.39 toward positive and negative infinity, you would see that each graph becomes arbitrarily close to the vertical line $x = 1$. This line is a **vertical asymptote** of the graph of f.

NOTE If a function f has a vertical asymptote at $x = c$, then f is *not continuous* at c.

Definition of Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of f.

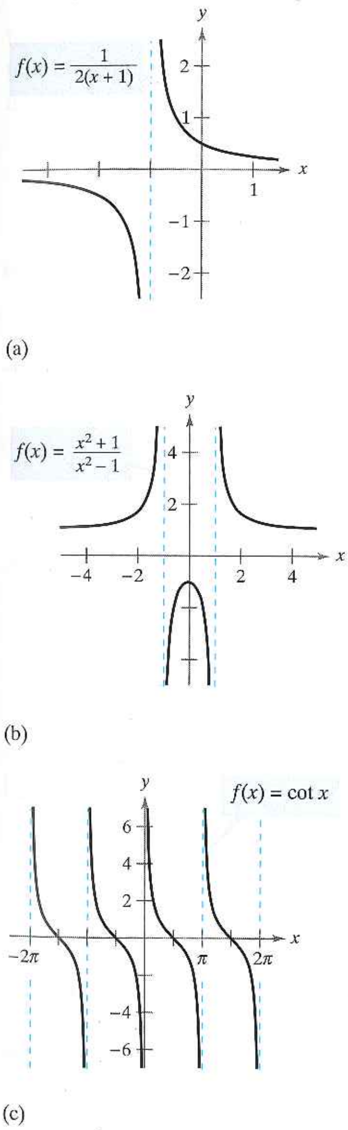
In Example 1, note that each of the functions is a *quotient* and that the vertical asymptote occurs at a number where the denominator is 0 (and the numerator is not 0). The next theorem generalizes this observation. (A proof of this theorem is given in the appendix.)

THEOREM 1.14 Vertical Asymptotes

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function given by

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.



Functions with vertical asymptotes
Figure 1.40

EXAMPLE 2 Finding Vertical Asymptotes

Determine all vertical asymptotes of the graph of each function.

a. $f(x) = \dfrac{1}{2(x+1)}$ **b.** $f(x) = \dfrac{x^2+1}{x^2-1}$ **c.** $f(x) = \cot x$

Solution

a. When $x = -1$, the denominator of

$$f(x) = \frac{1}{2(x+1)}$$

is 0 and the numerator is not 0. Hence, by Theorem 1.14, you can conclude that $x = -1$ is a vertical asymptote, as shown in Figure 1.40(a).

b. By factoring the denominator as

$$f(x) = \frac{x^2+1}{x^2-1} = \frac{x^2+1}{(x-1)(x+1)}$$

you can see that the denominator is 0 at $x = -1$ and $x = 1$. Moreover, because the numerator is not 0 at these two points, you can apply Theorem 1.14 to conclude that the graph of f has two vertical asymptotes, as shown in Figure 1.40(b).

c. By writing the cotangent function in the form

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

you can apply Theorem 1.14 to conclude that vertical asymptotes occur at all values of x such that $\cos x \neq 0$ and $\sin x = 0$, as shown in Figure 1.40(c). Hence, the graph of this function has infinitely many vertical asymptotes. These asymptotes occur when $x = n\pi$, where n is an integer.

Theorem 1.14 requires that the value of the numerator at $x = c$ be nonzero. If both the numerator and the denominator are 0 at $x = c$, you obtain the *indeterminate form* $0/0$, and you cannot determine the limit behavior at $x = c$ without further investigation.

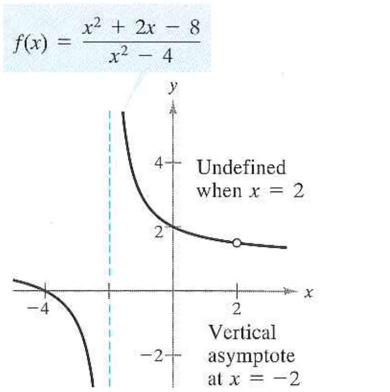
EXAMPLE 3 A Rational Function with Common Factors

Determine all vertical asymptotes of the graph of

$$f(x) = \frac{x^2 + 2x - 8}{x^2 - 4}.$$

Solution Begin by simplifying the expression, as follows.

$$\begin{aligned} f(x) &= \frac{x^2 + 2x - 8}{x^2 - 4} \\ &= \frac{(x + 4)(x - 2)}{(x + 2)(x - 2)} \\ &= \frac{x + 4}{x + 2}, \quad x \neq 2 \end{aligned}$$



$f(x)$ increases and decreases without bound as x approaches -2.
Figure 1.41

At all x-values other than $x = 2$, the graph of f coincides with the graph of $g(x) = (x + 4)/(x + 2)$. Thus, you can apply Theorem 1.14 to g to conclude that there is a vertical asymptote at $x = -2$, as shown in Figure 1.41. From the graph, you can see that

$$\lim_{x\to -2^-} \frac{x^2 + 2x - 8}{x^2 - 4} = -\infty \quad \text{and} \quad \lim_{x\to -2^+} \frac{x^2 + 2x - 8}{x^2 - 4} = \infty.$$

Note that $x = 2$ is *not* a vertical asymptote.

EXAMPLE 4 Determining Infinite Limits

Find each limit.

$$\lim_{x\to 1^-} \frac{x^2 - 3x}{x - 1} \quad \text{and} \quad \lim_{x\to 1^+} \frac{x^2 - 3x}{x - 1}$$

Solution Because the denominator is 0 when $x = 1$ (and the numerator is not zero), you know that the graph of

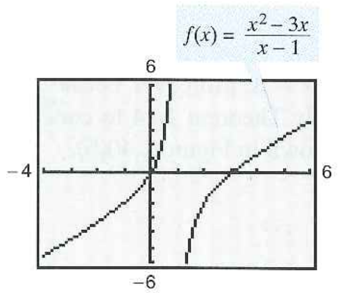
$$f(x) = \frac{x^2 - 3x}{x - 1}$$

has a vertical asymptote at $x = 1$. This means that each of the given limits is either ∞ or $-\infty$. A graphing utility can help determine the result. From the graph of f shown in Figure 1.42, you can see that the graph approaches ∞ from the left of $x = 1$ and approaches $-\infty$ from the right of $x = 1$. Thus, you can conclude that

$$\lim_{x\to 1^-} \frac{x^2 - 3x}{x - 1} = \infty \qquad \text{The limit from the left is infinity.}$$

and

$$\lim_{x\to 1^+} \frac{x^2 - 3x}{x - 1} = -\infty. \qquad \text{The limit from the right is negative infinity.}$$



f has a vertical asymptote at $x = 1$.
Figure 1.42

STUDY TIP When using a graphing calculator or graphing software, be careful to correctly interpret the graph of a function with a vertical asymptote—graphing utilities often have difficulty drawing this type of graph.

THEOREM 1.15 Properties of Infinite Limits

Let c and L be real numbers and let f and g be functions such that

$$\lim_{x\to c} f(x) = \infty \quad \text{and} \quad \lim_{x\to c} g(x) = L.$$

1. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = \infty$

2. Product: $\lim_{x\to c} [f(x)g(x)] = \infty, \quad L > 0$

$\lim_{x\to c} [f(x)g(x)] = -\infty, \quad L < 0$

3. Quotient: $\lim_{x\to c} \dfrac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$.

Proof To show that the limit of $f(x) + g(x)$ is infinite, choose $M > 0$. You then need to find $\delta > 0$ such that

$$[f(x) + g(x)] > M$$

whenever $0 < |x - c| < \delta$. For simplicity's sake, you can assume L is positive and let $M_1 = M + 1$. Because the limit of $f(x)$ is infinite, there exists δ_1 such that $f(x) > M_1$ whenever $0 < |x - c| < \delta_1$. Also because the limit of $g(x)$ is L, there exists δ_2 such that $|g(x) - L| < 1$ whenever $0 < |x - c| < \delta_2$. By letting δ be the smaller of δ_1 and δ_2, you can conclude that $0 < |x - c| < \delta$ implies $f(x) > M + 1$ and $|g(x) - L| < 1$. The second of these two inequalities implies that $g(x) > L - 1$, and, adding this to the first inequality, you can write

$$f(x) + g(x) > (M + 1) + (L - 1) = M + L > M.$$

Thus, you can conclude that

$$\lim_{x\to c} [f(x) + g(x)] = \infty.$$

We leave the proofs of the remaining properties as exercises (see Exercise 60).

EXAMPLE 5 Determining Limits

a. Because $\lim_{x\to 0} 1 = 1$ and $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$, you can write

$$\lim_{x\to 0} \left(1 + \frac{1}{x^2}\right) = \infty. \qquad \text{Property 1, Theorem 1.15}$$

b. Because $\lim_{x\to 1^-} (x^2 + 1) = 2$ and $\lim_{x\to 1^-} \dfrac{1}{x - 1} = -\infty$, you can write

$$\lim_{x\to 1^-} \frac{x^2 + 1}{1/(x - 1)} = 0. \qquad \text{Property 3, Theorem 1.15}$$

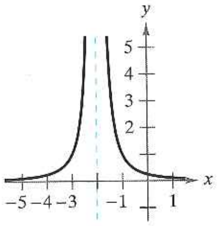
c. Because $\lim_{x\to 0^+} 3 = 3$ and $\lim_{x\to 0^+} \cot x = \infty$, you can write

$$\lim_{x\to 0^+} 3 \cot x = \infty. \qquad \text{Property 2, Theorem 1.15}$$

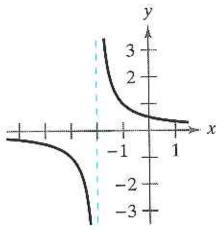
EXERCISES FOR SECTION 1.5

In Exercises 1–4, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -2 from the left and from the right.

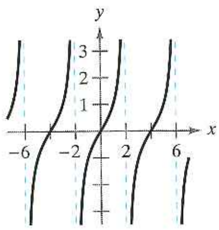
1. $f(x) = \dfrac{1}{(x+2)^2}$



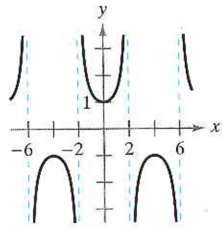
2. $f(x) = \dfrac{1}{x+2}$



3. $f(x) = \tan \dfrac{\pi x}{4}$



4. $f(x) = \sec \dfrac{\pi x}{4}$



Numerical and Graphical Analysis **In Exercises 5–8, determine whether $f(x)$ approaches ∞ or $-\infty$ as x approaches -3 from the left and from the right by completing the table. Use a graphing utility to graph the function and confirm your answer.**

x	-3.5	-3.1	-3.01	-3.001
$f(x)$				

x	-2.999	-2.99	-2.9	-2.5
$f(x)$				

5. $f(x) = \dfrac{1}{x^2 - 9}$

6. $f(x) = \dfrac{x}{x^2 - 9}$

7. $f(x) = \dfrac{x^2}{x^2 - 9}$

8. $f(x) = \sec \dfrac{\pi x}{6}$

In Exercises 9–24, find the vertical asymptotes (if any) of the function.

9. $f(x) = \dfrac{1}{x^2}$

10. $f(x) = \dfrac{4}{(x-2)^3}$

11. $h(x) = \dfrac{x^2 - 2}{x^2 - x - 2}$

12. $g(x) = \dfrac{2 + x}{1 - x}$

13. $f(x) = \dfrac{x^3}{x^2 - 1}$

14. $f(x) = \dfrac{-4x}{x^2 + 4}$

15. $f(x) = \tan 2x$

16. $f(x) = \sec \pi x$

17. $T(t) = 1 - \dfrac{4}{t^2}$

18. $V(s) = \dfrac{-2}{(s-2)^2}$

19. $f(x) = \dfrac{x}{x^2 + x - 2}$

20. $f(x) = \dfrac{1}{(x+3)^4}$

21. $g(x) = \dfrac{x^3 + 1}{x + 1}$

22. $h(x) = \dfrac{x^2 - 4}{x^3 + 2x^2 + x + 2}$

23. $s(t) = \dfrac{t}{\sin t}$

24. $g(\theta) = \dfrac{\tan \theta}{\theta}$

In Exercises 25–28, determine whether the function has a vertical asymptote or a removable discontinuity at $x = -1$. Graph the function using a graphing utility to confirm your answer.

25. $f(x) = \dfrac{x^2 - 1}{x + 1}$

26. $f(x) = \dfrac{x^2 - 6x - 7}{x + 1}$

27. $f(x) = \dfrac{x^2 + 1}{x + 1}$

28. $f(x) = \dfrac{\sin(x + 1)}{x + 1}$

In Exercises 29–40, find the limit.

29. $\displaystyle\lim_{x\to 2^+} \frac{x-3}{x-2}$

30. $\displaystyle\lim_{x\to 1^+} \frac{2+x}{1-x}$

31. $\displaystyle\lim_{x\to 4^+} \frac{x^2}{x^2-16}$

32. $\displaystyle\lim_{x\to 4^-} \frac{x^2}{x^2+16}$

33. $\displaystyle\lim_{x\to -3^-} \frac{x^2+2x-3}{x^2+x-6}$

34. $\displaystyle\lim_{x\to(-1/2)^+} \frac{6x^2+x-1}{4x^2-4x-3}$

35. $\displaystyle\lim_{x\to 0^-} \left(1 + \frac{1}{x}\right)$

36. $\displaystyle\lim_{x\to 0^-} \left(x^2 - \frac{1}{x}\right)$

37. $\displaystyle\lim_{x\to 0^+} \frac{2}{\sin x}$

38. $\displaystyle\lim_{x\to(\pi/2)^+} \frac{-2}{\cos x}$

39. $\displaystyle\lim_{x\to 1} \frac{x^2-x}{(x^2+1)(x-1)}$

40. $\displaystyle\lim_{x\to 3} \frac{x-2}{x^2}$

In Exercises 41–44, use a graphing utility to graph the function and determine the one-sided limit.

41. $f(x) = \dfrac{x^2 + x + 1}{x^3 - 1}$

$\displaystyle\lim_{x\to 1^+} f(x)$

42. $f(x) = \dfrac{x^3 - 1}{x^2 + x + 1}$

$\displaystyle\lim_{x\to 1^-} f(x)$

43. $f(x) = \dfrac{1}{x^2 - 25}$

$\displaystyle\lim_{x\to 5^-} f(x)$

44. $f(x) = \sec \dfrac{\pi x}{6}$

$\displaystyle\lim_{x\to 3^+} f(x)$

45. A given sum S is inversely proportional to $1 - r$, where $0 < |r| < 1$. Find the limit of S as $r \to 1^-$.

46. ***Boyle's Law*** For a quantity of gas at a constant temperature, the pressure P is inversely proportional to the volume V. Find the limit of P as $V \to 0^+$.

47. *Rate of Change* A 25-foot ladder is leaning against a house (see figure). If the base of the ladder is pulled away from the house at a rate of 2 feet per second, the top will move down the wall at a rate of

$$r = \frac{2x}{\sqrt{625 - x^2}} \text{ ft/sec}$$

where x is the distance between the base of the ladder and the house.

(a) Find the rate r when x is 7 feet.

(b) Find the rate r when x is 15 feet.

(c) Find the limit of r as $x \to 25^-$.

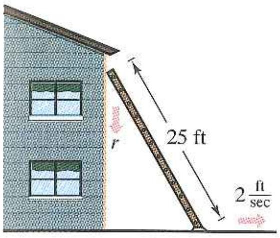


Figure for 47

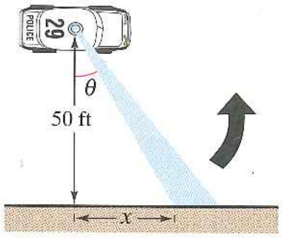


Figure for 48

48. *Rate of Change* A patrol car is parked 50 feet from a long warehouse (see figure). The revolving light on top of the car turns at a rate of $\frac{1}{2}$ revolution per second. The rate at which the light beam moves along the wall is

$$r = 50\pi \sec^2 \theta \text{ ft/sec.}$$

(a) Find the rate r when θ is $\pi/6$.

(b) Find the rate r when θ is $\pi/3$.

(c) Find the limit of r as $\theta \to (\pi/2)^-$.

49. *Illegal Drugs* The cost in millions of dollars for a governmental agency to seize $x\%$ of an illegal drug is

$$C = \frac{528x}{100 - x}, \quad 0 \le x < 100.$$

(a) Find the cost of seizing 25%.

(b) Find the cost of seizing 50%.

(c) Find the cost of seizing 75%.

(d) Find the limit of C as $x \to 100^-$.

50. *Average Speed* On a trip of d miles to another city, a truck driver's average speed was x miles per hour. On the return trip the average speed was y miles per hour. The average speed for the round trip was 50 miles per hour.

(a) Verify that $y = \dfrac{25x}{x - 25}$. What is the domain?

(b) Complete the table.

x	30	40	50	60
y				

Are the values of y different than expected? Explain.

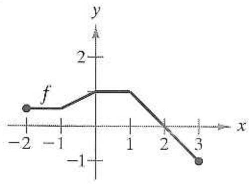
(c) Find the limit of y as $x \to 25^+$.

51. *Relativity* According to the theory of relativity, the mass m of a particle depends on its velocity v. That is,

$$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}$$

where m_0 is the mass when the particle is at rest and c is the speed of light. Find the limit of the mass as v approaches c^-.

52. *Think About It* Use the graph of the function f (see figure) to sketch the graph of $g(x) = 1/f(x)$ on the interval $[-2, 3]$.



53. *Numerical and Graphical Reasoning* A crossed belt connects a 20-centimeter pulley (10-cm radius) on an electric motor with a 40-centimeter pulley (20-cm radius) on a saw arbor (see figure). The electric motor runs at 1700 revolutions per minute.

(a) Determine the number of revolutions per minute of the saw.

(b) How does crossing the belt affect the saw in relation to the motor?

(c) Let L be the total length of the belt. Write L as a function of ϕ, where ϕ is measured in radians. What is the domain of the function? [*Hint:* Add the lengths of the straight sections of the belt and the length of the belt around each pulley.]

(d) Use a graphing utility to complete the table.

ϕ	0.3	0.6	0.9	1.2	1.5
L					

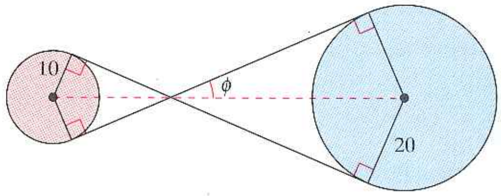
(e) Use a graphing utility to graph the function over the appropriate domain.

(f) Find

$$\lim_{\phi \to (\pi/2)^-} L.$$

Use a geometric argument as the basis of a second method of finding this limit.

(g) Find $\lim_{\phi \to 0^+} L$.



54. ***Numerical and Graphical Analysis*** Consider the shaded region outside the sector of a circle of radius 10 meters and inside a right triangle (see figure).

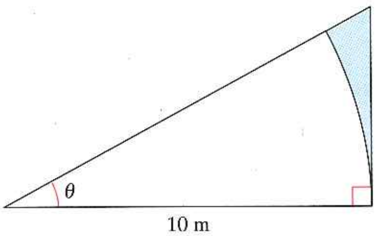
(a) Write the area $A = f(\theta)$ of the region as a function of θ. Determine the domain of the function.

(b) Use a graphing utility to complete the table.

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$					

(c) Use a graphing utility to graph the function over the appropriate domain.

(d) Find the limit of A as $\theta \to \pi/2^-$.



True or False? **In Exercises 55–58, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

55. If $p(x)$ is a polynomial, then the function given by

$$f(x) = \frac{p(x)}{x-1}$$

has a vertical asymptote at $x = 1$.

56. A rational function has at least one vertical asymptote.

57. Polynomial functions have no vertical asymptotes.

58. If f has a vertical asymptote at $x = 0$, then f is undefined at $x = 0$.

59. Find functions f and g such that

$$\lim_{x\to c} f(x) = \infty \quad \text{and} \quad \lim_{x\to c} g(x) = \infty$$

but $\lim_{x\to c} [f(x) - g(x)] \neq 0$.

60. Prove the remaining properties of Theorem 1.15.

61. Prove that if

$$\lim_{x\to c} f(x) = \infty$$

then

$$\lim_{x\to c} \frac{1}{f(x)} = 0.$$

62. Prove that if

$$\lim_{x\to c} \frac{1}{f(x)} = 0$$

then $\lim_{x\to c} f(x)$ does not exist.

63. ***Writing*** Because

$$\lim_{x\to 1} \frac{x+2}{(x-1)^2} = \infty,$$

for each $M > 0$ there exists a $\delta > 0$ such that

$$\frac{x+2}{(x-1)^2} > M$$

whenever $0 < |x-1| < \delta$.

(a) Let $M = 100$. Use a graphing utility to graph the function and the line $y = 100$. Use the graphs to estimate δ.

(b) Let $M = 1000$. Use a graphing utility to graph the function and the line $y = 1000$. Use the graphs to estimate δ.

(c) Write a short paragraph describing the change in δ for increasing M.

SECTION PROJECT

Recall from Theorem 1.9 that the limit of $f(x) = \dfrac{\sin x}{x}$ as x approaches 0 is

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

(a) Use a graphing utility to graph the function f on the interval $-\pi \le 0 \le \pi$. Explain how this graph helps confirm that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

(b) Explain how you could use a table of values to confirm the value of this limit numerically.

(c) Graph the function $g(x) = \sin x$ by hand. Sketch a tangent line at the point $(0, 0)$ and visually estimate the slope of this tangent line.

(d) Let $(x, \sin x)$ be a point on the graph of g near $(0, 0)$, and write a formula for the slope of the secant line joining $(x, \sin x)$ and $(0, 0)$. Evaluate this formula for $x = 0.1$ and $x = 0.01$. Then determine the exact slope of the tangent line to g at the point $(0, 0)$.

(e) Sketch the graph of the cosine function $h(x) = \cos x$. What is the slope of the tangent line at the point $(0, 1)$? Use limits to find this slope analytically. [*Hint:* See the second part of Theorem 1.9.]

(f) Find the slope of the tangent line to $k(x) = \tan x$ at the point $(0, 0)$.