73. $R = vt\cos\alpha$

$R = v\left(\frac{2v\sin\alpha}{g}\right)\cos\alpha$

$R = \frac{2v^2\sin\alpha\cos\alpha}{g}$

$R = \frac{v^2\sin 2\alpha}{g}$

75. $2\left(\sin^2\frac{1}{2}\alpha - \sin^2\frac{1}{2}\theta\right)$

$= 2\left(\frac{1-\cos 2\left(\frac{1}{2}\alpha\right)}{2} - \frac{1-\cos 2\left(\frac{1}{2}\theta\right)}{2}\right)$

$= 2\left(\frac{1-\cos\alpha - 1 + \cos\theta}{2}\right)$

$= \cos\theta - \cos\alpha$

Section 9.4 **Using Trigonometric Identities**

Example Notes

For **Example 3**, an expression for $2x$ that is equivalent to $\frac{5\pi}{3} + 2\pi k$ is $-\frac{\pi}{3} + 2\pi k$.

The corresponding expression for x that is equivalent to $\frac{5\pi}{6} + \pi k$ is $-\frac{\pi}{6} + \pi k$.

These alternate expressions would be the result of applying the method shown in the table on page 540.

Take this opportunity to remind students that different expressions can represent the same set of numbers.

$\frac{5\pi}{6} + \pi k$ represents

$\ldots, -\frac{7\pi}{6}, -\frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}, \ldots$

$-\frac{\pi}{6} + \pi k$ represents the same set:

$\ldots, -\frac{7\pi}{6}, -\frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}, \ldots$

73. The horizontal range of a projectile R is given by the equation

$$R = vt\cos\alpha,$$

where v is the initial velocity of the projectile, t is the time of flight, and α is the angle between the line of fire and the horizontal. If $t = \frac{2v\sin\alpha}{g}$, where g is acceleration due to gravity, show that $R = \frac{v^2\sin 2\alpha}{g}$.

74. The expression $\sin\left(\frac{\pi}{2} - 2\theta\right)$ occurs in the theory of reflection of light waves. Show that this expression can be written as $1 - 2\sin^2\theta$.

$\sin\left(\frac{\pi}{2} - 2\theta\right)$

$= \sin\frac{\pi}{2}\cos 2\theta - \cos\frac{\pi}{2}\sin 2\theta$

$= 1\cdot\cos 2\theta - 0\cdot\sin 2\theta$

$= \cos 2\theta$

$= 1 - 2\sin^2\theta$

75. The expression $2\left(\sin^2\frac{1}{2}\alpha - \sin^2\frac{1}{2}\theta\right)$ is used in the theory of the motion of a pendulum. Show that this equation can be written as $\cos\theta - \cos\alpha$.

76. A batter hits a baseball that is caught by a fielder. If the ball leaves the bat at an angle of θ radians to the horizontal, with an initial velocity of v feet per second, then the approximate horizontal distance d traveled by the ball is given by

$$d = \frac{v^2\sin\theta\cos\theta}{16}.$$

a. If the initial velocity is 90 ft/sec, find the horizontal distance traveled by the ball when $\theta = 0.5$ radian and when $\theta = 0.75$ radian.

b. Use an identity to show that $d = \frac{v^2\sin 2\theta}{32}$.

a. 212.9973 ft; 252.4909 ft

b. $d = \frac{v^2\sin\theta\cos\theta}{16} = \frac{2}{2}\left(\frac{v^2\sin\theta\cos\theta}{16}\right)$

$= \frac{v^2 2\sin\theta\cos\theta}{32}$

$= \frac{v^2\sin 2\theta}{32}$

9.4 Using Trigonometric Identities

Objectives

- Use identities to solve trigonometric equations

Recall that the basic identities are used to simplify expressions and to algebraically solve trigonometric equations. The trigonometric identities introduced in Section 9.3 can also be used with the techniques shown in Section 8.3, where equations were rewritten into a basic form and then solved.

Example 1 Use Double-Angle Identities

Solve $5\cos x + 3\cos 2x = 3$.

Solution

Use a double-angle identity to rewrite $\cos 2x$ in terms of $\cos^2 x$.

$$5\cos x + 3\cos 2x = 3$$

$$5\cos x + 3(2\cos^2 x - 1) = 3 \quad \textit{double-angle identity}$$

$$5\cos x + 6\cos^2 x - 3 = 3$$

$$6\cos^2 x + 5\cos x - 6 = 0$$

$$(2\cos x + 3)(3\cos x - 2) = 0 \quad \textit{factor the quadratic expression}$$

$$2\cos x + 3 = 0 \quad \text{or} \quad 3\cos x - 2 = 0$$

$$\cos x = -\frac{3}{2} \qquad \cos x = \frac{2}{3}$$

The equation $\cos x = -\frac{3}{2}$ has no solutions because $\cos x$ always lies between -1 and 1. A calculator shows that the solutions of $\cos x = \frac{2}{3}$ are

$$x \approx 0.8411 + 2k\pi \quad \text{and} \quad x \approx -0.8411 + 2k\pi$$

for any integer k.

Example 2 Use Double-Angle Identities

Solve the equation $\sin x \cos x = 1$.

Solution

Use the double-angle identity to rewrite $\sin x \cos x$.

$$2 \sin x \cos x = \sin 2x$$
$$\sin x \cos x = \frac{1}{2} \sin 2x$$

Replace $\sin x \cos x$ with $\frac{1}{2} \sin 2x$ and multiply both sides by 2.

$$\frac{1}{2} \sin 2x = 1$$
$$\sin 2x = 2$$

Because the sine of any number must be between -1 and 1, there is no solution to the last equation. Therefore, there is no solution to the original equation.

Example 3 Use Double-Angle Identities

Find exact solutions of $\cos^2 x - \sin^2 x = \frac{1}{2}$.

Solution

Because $\cos^2 x - \sin^2 x = \cos 2x$, the equation can be rewritten

$$\cos 2x = \frac{1}{2}$$
$$2x = \cos^{-1} \frac{1}{2}$$

$$2x = \frac{\pi}{3} + 2\pi k \quad \text{or} \quad 2x = \frac{5\pi}{3} + 2\pi k$$
$$x = \frac{\pi}{6} + \pi k \qquad\qquad x = \frac{5\pi}{6} + \pi k$$

for any integer k.

ADDITIONAL EXAMPLES

Example 1

Solve $\cos 2x + 5 \sin x = 3$.

Use $\cos 2x = 1 - 2 \sin^2 x$.

$$\cos 2x + 5 \sin x = 3$$
$$(1 - 2 \sin^2 x) + 5 \sin x = 3$$
$$-2 \sin^2 x + 5 \sin x - 2 = 0$$
$$2 \sin^2 x - 5 \sin x + 2 = 0$$
$$(2 \sin x - 1)(\sin x - 2) = 0$$
$$2 \sin x - 1 = 0 \text{ or } \sin x - 2 = 0$$
$$\sin x = \frac{1}{2} \text{ or } \sin x = 2$$

(no solution)

$$x = \frac{\pi}{6} + 2k\pi \text{ or}$$
$$x = \pi - \frac{\pi}{6} + 2k\pi = \frac{5\pi}{6} + 2k\pi$$

So, all solutions of the original equation are:

$x = \frac{\pi}{6} + 2k\pi$ and $x = \frac{5\pi}{6} + 2k\pi$ for any integer k.

Example 2

Solve the equation $8 \sin x \cos x = 5$.

$$8 \sin x \cos x = 5$$
$$4(2 \sin x \cos x) = 5$$
$$2 \sin x \cos x = \frac{5}{4}$$
$$\sin 2x = \frac{5}{4}$$

There is no solution to $\sin 2x = \frac{5}{4}$, so there is no solution to $8 \sin x \cos x = 5$.

Example 3

Find exact solutions of $2 \sin x \cos x = \frac{1}{2}$.

$$2 \sin x \cos x = \frac{1}{2}$$
$$\sin 2x = \frac{1}{2}$$

$$2x = \frac{\pi}{6} + 2\pi k \quad \text{or} \quad 2x = \frac{5\pi}{6} + 2\pi k$$
$$x = \frac{\pi}{12} + \pi k \qquad\qquad x = \frac{5\pi}{12} + \pi k$$

Example Notes

In **Example 5,** make sure students understand how to find all the solutions after the statement $\cos x = -\frac{1}{2}$ or $\cos x = 1$.

If $\cos x = -\frac{1}{2}$, then (using the table on page 540), $x = \frac{2\pi}{3} + 2\pi k$ or $x = -\frac{2\pi}{3} + 2\pi k$. Because $-\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ are coterminal, these solutions can be written as $x = \frac{2\pi}{3} + 2\pi k$ or $x = \frac{4\pi}{3} + 2\pi k$.

If $\cos x = 1$, then $x = 0 + 2\pi k$.

Now find all solutions where $0 \le x \le 2\pi$:

Let $k = 0$ in $x = \frac{2\pi}{3} + 2\pi k$; $x = \frac{2\pi}{3}$

Let $k = 0$ in $x = \frac{4\pi}{3} + 2\pi k$; $x = \frac{4\pi}{3}$

Let $k = 0$ in $x = 0 + 2\pi k$; $x = 0$

Let $k = 1$ in $x = 0 + 2\pi k$; $x = 2\pi$

Have students check *all* solutions for Example 5 as instructed in the **CAUTION** on this page.

Students should find that 0, $\frac{2\pi}{3}$, and 2π are solutions, and that $\frac{4\pi}{3}$ is not. The checks for $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ are shown below.

Check $x = \frac{2\pi}{3}$: $\sin x = \sin\frac{x}{2}$

$$\sin\frac{2\pi}{3} \;\Big|\; \sin\frac{\frac{2\pi}{3}}{2}$$
$$\;\Big|\; \sin\frac{\pi}{3}$$
$$\frac{\sqrt{3}}{2} \;\Big|\; \frac{\sqrt{3}}{2} \quad ✓$$

$\frac{2\pi}{3}$ is a solution.

Check $x = \frac{4\pi}{3}$: $\sin x = \sin\frac{x}{2}$

$$\sin\frac{4\pi}{3} \;\Big|\; \sin\frac{\frac{4\pi}{3}}{2}$$
$$\;\Big|\; \sin\frac{2\pi}{3}$$
$$-\frac{\sqrt{3}}{2} \;\Big|\; \frac{\sqrt{3}}{2} \quad ✗$$

$\frac{4\pi}{3}$ is not a solution.

Example 4 Use Addition Identities

Find the exact solutions of $\sin 2x \cos x + \cos 2x \sin x = 1$.

Solution

The left side of the equation is similar to the right side of the addition identity for sine.

$$\sin(x + y) = \sin x \cos y + \cos x \sin y.$$

Substitute $2x$ for x and x for y.

$$\begin{aligned} \sin 2x \cos x + \cos 2x \sin x = \sin(2x + x) &= 1 \\ \sin 3x &= 1 \\ 3x &= \sin^{-1} 1 \end{aligned}$$

For any integer k,

$$\begin{aligned} 3x &= \frac{\pi}{2} + 2\pi k \\ x &= \frac{\pi}{6} + \frac{2\pi k}{3} \end{aligned}$$

Example 5 Use Half-Angle Identities

Find the solutions of $\sin x = \sin\frac{x}{2}$, where $0 \le x \le 2\pi$.

Solution

$$\begin{aligned} \sin x &= \sin\frac{x}{2} \\ \sin x &= \pm\sqrt{\frac{1 - \cos x}{2}} && \textit{half-angle identity} \\ \sin^2 x &= \frac{1 - \cos x}{2} && \textit{square both sides} \\ 1 - \cos^2 x &= \frac{1 - \cos x}{2} && \textit{Pythagorean identity} \\ 2 - 2\cos^2 x &= 1 - \cos x \\ 2\cos^2 x - \cos x - 1 &= 0 \\ (2\cos x + 1)(\cos x - 1) &= 0 \end{aligned}$$

$$2\cos x + 1 = 0 \quad \text{or} \quad \cos x - 1 = 0$$

$$\cos x = -\frac{1}{2} \qquad\qquad \cos x = 1$$

For any integer k,

$$x = \frac{2\pi}{3} + 2\pi k \qquad x = \frac{4\pi}{3} + 2\pi k \qquad x = 0 + 2\pi k$$

For $0 \le x \le 2\pi$, $x = 0, \frac{2\pi}{3}$, or 2π.

CAUTION

Squaring both sides of an equation may introduce extraneous solutions. Be sure to check all solutions in the original equation.

Solving $a \sin x + b \cos x = c$ (Optional)

Equations of the form $a \sin x + b \cos x = c$ occur often.

For the case when $c = 0$, the equations can be rewritten as

$$a \sin x = -b \cos x$$
$$\frac{\sin x}{\cos x} = -\frac{b}{a}$$
$$\tan x = -\frac{b}{a}$$

The last equation can be solved by methods discussed in Section 8.3.

For the case when $c \neq 0$, a very different approach is needed to find the solutions to

$$a \sin x + b \cos x = c$$

The procedure involves rewriting the equation as

$$\sin (x + \alpha) = k,$$

where α is the angle whose terminal side contains the point (a, b), and then using the addition identity for the sine function.

To find α, construct a right triangle in the coordinate plane with sides a and b, where a lies on the positive x-axis and α is the angle with its vertex at the origin.

Begin by writing the equation so that the coefficient of $\sin x$, a, is positive. The position of the point (a, b) depends on whether b is positive or negative. If b is positive, the point is in the first quadrant; if b is negative, the point is in the fourth quadrant. Both possibilities are shown in Figures 9.4-1 and 9.4-2.

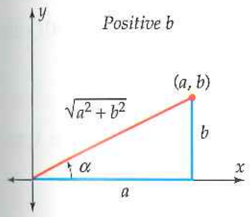


Figure 9.4-1

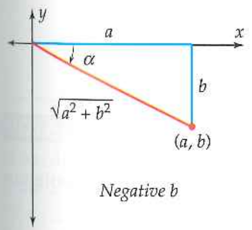


Figure 9.4-2

In both cases, $\sin \alpha = \dfrac{b}{\sqrt{a^2 + b^2}}$ and $\cos \alpha = \dfrac{a}{\sqrt{a^2 + b^2}}$. [1]

Divide each side of the original equation by $\sqrt{a^2 + b^2}$ to obtain

$$a \sin x \quad + \quad b \cos x \quad = \quad c$$
$$\frac{a}{\sqrt{a^2 + b^2}} \sin x + \frac{b}{\sqrt{a^2 + b^2}} \cos x = \frac{c}{\sqrt{a^2 + b^2}} \qquad [2]$$

Substitute the equivalent expressions from [1] into equation [2].

$$\cos \alpha \sin x + \sin \alpha \cos x = \frac{c}{\sqrt{a^2 + b^2}}$$

Use the addition identity for sine to rewrite the left side of the equation.

$$\sin(x + \alpha) = \frac{c}{\sqrt{a^2 + b^2}}$$

The last equation can be solved by using the methods from Section 8.3. The steps of the procedure are summarized in the following box.

COMMON ERROR ALERT

In **Example 4,** students might incorrectly solve $3x = \frac{\pi}{2} + 2\pi k$, getting $x = \frac{\pi}{6} + 2\pi k$. Remind them to divide *both* terms on the right side of the equation by 3 to get $x = \frac{\pi}{6} + \frac{2\pi k}{3}$.

ADDITIONAL EXAMPLES

Example 4

Find the exact solutions of $\cos 2x \cos x - \sin 2x \sin x = -1$.

$$\cos 2x \cos x - \sin 2x \sin x = -1$$
$$\cos(2x + x) = -1$$
$$\cos 3x = -1$$
$$3x = \pi + 2\pi k$$
$$x = \frac{\pi}{3} + \frac{2\pi k}{3}$$

for any integer k.

Example 5

Find the solutions of $\sin x = \cos \frac{x}{2}$, where $0 \leq x \leq 2\pi$.

$$\sin x = \cos \frac{x}{2}$$
$$\sin x = \pm\sqrt{\frac{1 + \cos x}{2}}$$
$$\sin^2 x = \frac{1 + \cos x}{2}$$
$$1 - \cos^2 x = \frac{1 + \cos x}{2}$$
$$2 - 2\cos^2 x = 1 + \cos x$$
$$2\cos^2 x + \cos x - 1 = 0$$
$$(2 \cos x - 1)(\cos x + 1) = 0$$
$$2 \cos x - 1 = 0 \quad \text{or} \quad \cos x + 1 = 0$$
$$\cos x = \frac{1}{2} \quad \text{or} \quad \cos x = -1$$

For any integer k,
$x = \frac{\pi}{3} + 2\pi k,\quad x = -\frac{\pi}{3} + 2\pi k,$
$x = \pi + 2\pi k$

For $0 \leq x \leq 2\pi$, $x = \frac{\pi}{3}, \pi$, or $\frac{5\pi}{3}$.

Checking shows that all 3 solutions satisfy the original equation.

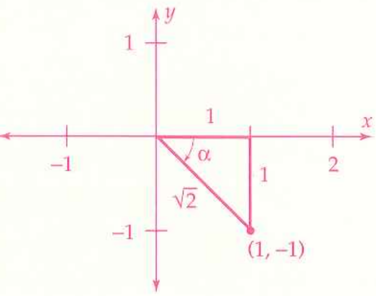
ADDITIONAL EXAMPLES

Example 6

Solve the equation
$-\sin x + \cos x = -1$.

1. Multiply both sides of the equation by -1.
$$-\sin x + \cos x = -1$$
$$\sin x - \cos x = 1$$
So, $a = 1$ and $b = -1$.

2. Sketch a diagram of the angle α that has $(1, -1)$ on its terminal side.



3. The length of the hypotenuse is $\sqrt{1^2 + (-1)^2} = \sqrt{2}$.
Find $\sin \alpha$ and $\cos \alpha$ from the figure.
$$\cos \alpha = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\sin \alpha = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$
Find α: $\alpha = -\frac{\pi}{4}$

4. Divide both sides of the equation in Step 1 by the hypotenuse, $\sqrt{2}$.
$$\sin x - \cos x = 1$$
$$\frac{\sin x}{\sqrt{2}} - \frac{\cos x}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$
$$\frac{\sqrt{2}}{2}\sin x - \frac{\sqrt{2}}{2}\cos x = \frac{\sqrt{2}}{2}$$

5. Rewrite the equation by substituting $\cos\left(-\frac{\pi}{4}\right)$ for $\frac{\sqrt{2}}{2}$ and $\sin\left(-\frac{\pi}{4}\right)$ for $-\frac{\sqrt{2}}{2}$. Then, use the addition identity for sine.
$$\frac{\sqrt{2}}{2}\sin x - \frac{\sqrt{2}}{2}\cos x = \frac{\sqrt{2}}{2}$$
$$\cos\left(-\frac{\pi}{4}\right)\sin x + \sin\left(-\frac{\pi}{4}\right)\cos x = \frac{\sqrt{2}}{2}$$
$$\sin\left(x + \left(-\frac{\pi}{4}\right)\right) = \frac{\sqrt{2}}{2}$$
$$\sin\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

Solving $a \sin x + b \cos x = c$, where $c \neq 0$

Let a, b, and c be nonzero real numbers. To solve
$$a \sin x + b \cos x = c$$

1. **Multiply by -1, if needed, to make a positive.**
2. **Plot the point (a, b) and let α be the angle in standard position that contains (a, b) on its terminal side.**
3. **Find**
   - **the length of the hypotenuse of the reference triangle**
   - **expressions that represent $\sin \alpha$ and $\cos \alpha$**
   - **the measure of α**
4. **Divide each side of the equation by $\sqrt{a^2 + b^2}$ yielding**
$$\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x = \frac{c}{\sqrt{a^2+b^2}}$$
5. **Use the addition identity for sine to rewrite the equation.**
$$\sin(x + \alpha) = \frac{c}{\sqrt{a^2+b^2}}$$
6. **Solve the equation using techniques previously discussed.**

Example 6 Solve $a \sin x + b \cos x = c$

Solve the equation $-\sqrt{3}\sin x + \cos x = -\sqrt{3}$.

Solution

Step 1 Make the coefficient of $\sin x$ positive by multiplying both sides of the equation by -1.
$$\sqrt{3}\sin x - \cos x = \sqrt{3} \qquad \text{so } a = \sqrt{3} \text{ and } b = -1$$

Step 2 Sketch a diagram of the angle α that has $\left(\sqrt{3}, -1\right)$ on its terminal side. See Figure 9.4-3.

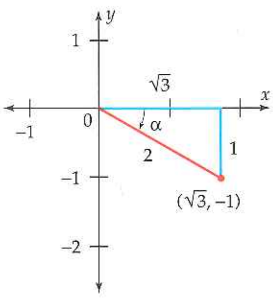


Figure 9.4-3

Step 3 The length of the hypotenuse is $\sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$.

Find $\sin \alpha$ and $\cos \alpha$ from the figure.
$$\cos \alpha = \frac{\sqrt{3}}{2} \text{ and } \sin \alpha = \frac{-1}{2}$$
Find α.
$$\alpha = -\frac{\pi}{6} \qquad \text{or} \qquad \alpha = \frac{11\pi}{6}$$

Step 4 Divide both sides of the equation in Step 1 by the hypotenuse, $\sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$.
$$\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x = \frac{\sqrt{3}}{2}$$

CAUTION
When substituting $-\sin\frac{11\pi}{6}$ for $\frac{1}{2}$, the sign between the terms changes to $+$.

Step 5 Rewrite the equation by substituting

$$\cos\frac{11\pi}{6} \text{ for } \frac{\sqrt{3}}{2} \quad \text{and} \quad -\sin\frac{11\pi}{6} \text{ for } \frac{1}{2}$$

and then use the addition identity for sine.

$$\cos\frac{11\pi}{6}\sin x + \sin\frac{11\pi}{6}\cos x = \frac{\sqrt{3}}{2}$$
$$\sin\left(x + \frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

Step 6 Solve the equation.

$$x + \frac{11\pi}{6} = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$x + \frac{11\pi}{6} = \frac{\pi}{3} + 2\pi k \quad \text{or} \quad x + \frac{11\pi}{6} = \frac{2\pi}{3} + 2\pi k$$
$$x = \frac{\pi}{3} - \frac{11\pi}{6} + 2\pi k \qquad x = \frac{2\pi}{3} - \frac{11\pi}{6} + 2\pi k$$
$$x = -\frac{3\pi}{2} + 2\pi k \qquad x = -\frac{7\pi}{6} + 2\pi k$$
$$x = \frac{\pi}{2} + 2\pi k \qquad x = \frac{5\pi}{6} + 2\pi k$$

for any integer k.

Maxima and Minima of $f(x) = a\sin x + b\cos x$

For functions of the form $f(x) = a\sin x + b\cos x$, maximum and minimum values can be found by using a technique similar to that described in the algorithm to solve equations of the form $a\sin x + b\cos x = c$.

Example 7 Maximum and Minimum of $f(x) = a\sin x + b\cos x$

Find the maximum and minimum of the function

$$f(x) = 3\sin x + 4\cos x.$$

Solution

Note that $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$. Let $\alpha = \cos^{-1}\frac{3}{5}$, or equivalently $\alpha = \sin^{-1}\frac{4}{5}$, and write the function in the form $f(x) = k\sin(x + \alpha)$.

$$f(x) = 5\left(\frac{3}{5}\sin x + \frac{4}{5}\cos x\right) = 5(\cos\alpha\sin x + \sin\alpha\cos x)$$
$$= 5\sin(x + \alpha)$$

Because the sine function varies between -1 and 1, the maximum of $5\sin(x + \alpha)$ is 5 and the minimum is -5.

NOTE The value of α is not needed to find the maximum or minimum of the function.

continued from page 606

6. Solve the equation.

$$\sin\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$x - \frac{\pi}{4} = \frac{\pi}{4} + 2\pi k \text{ or } x - \frac{\pi}{4} = \frac{3\pi}{4} + 2\pi k$$
$$x = \frac{\pi}{2} + 2\pi k \text{ or } x = \pi + 2\pi k$$

for any integer k.

Example 7

Find the maximum and minimum of the function $f(x) = 5\sin x + 12\cos x$.

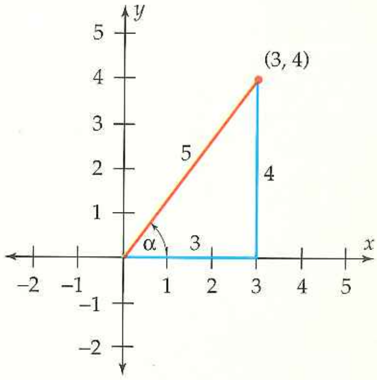
Note that $\sqrt{5^2 + 12^2} = \sqrt{169} = 13$.
Let $\alpha = \cos^{-1}\frac{5}{13} = \sin^{-1}\frac{12}{13}$.

$$f(x) = 13\left(\frac{5}{13}\sin x + \frac{12}{13}\cos x\right)$$
$$= 13(\cos\alpha\sin x + \sin\alpha\cos x)$$
$$= 13\sin(x + \alpha)$$

The maximum is 13 and the minimum is -13.

Example Notes

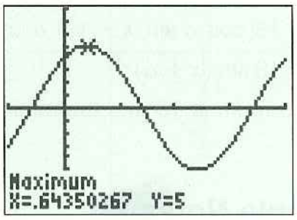
For **Example 7**, a diagram of the angle α might be helpful, as shown below.



Teaching Notes

Refer students to the margin NOTE at the bottom of page 607. As noted, the value of α is not needed to find the maximum or minimum of the function. However, the value of α *is needed to find the values of x that produce the maximum or minimum* of the function.

As shown at the right, the maximum value 5 for the function $f(x) = 3\sin x + 4\cos x$ occurs at $x \approx 0.6435$. So, (0.6435, 5) is a local maximum point on the graph, as shown below.



Exercises 9.4

ANSWERS

1. no solution
2. $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$
3. $x = 0, \pi, 2\pi, \frac{7\pi}{6}, \frac{11\pi}{6}$
4. $x = 0, 2\pi, \frac{\pi}{3}, \frac{5\pi}{3}$
5. $x = \frac{\pi}{2}, \frac{3\pi}{2}$
6. $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
7. $x = 0, 2\pi$
8. $t = 0, \pi, 2\pi$
9. $x = \frac{\pi}{2}, \frac{3\pi}{2}$
10. $x = 0, \frac{4\pi}{3}$
11. $x = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$
12. $x = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}, \frac{11\pi}{10}, \frac{13\pi}{10}, \frac{3\pi}{2}, \frac{17\pi}{10}, \frac{19\pi}{10}$
13. $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$
14. $x = \frac{\pi}{2}, \frac{3\pi}{2}$

To find the values of x that produce the maximum or minimum values of the function, solve the equation

$$a\sin x + b\cos x = c,$$

where c is the maximum or minimum value. In Example 7, the maximum value of 5 occurs when

$$3\sin x + 4\cos x = 5$$
$$\frac{3}{5}\sin x + \frac{4}{5}\cos x = 1$$
$$\sin(x + \alpha) = 1$$
$$x + \alpha = \sin^{-1} 1 = \frac{\pi}{2}$$
$$x = \frac{\pi}{2} - \alpha$$
$$\approx 1.5708 - 0.9273 \qquad \alpha = \cos^{-1}\frac{3}{5} \approx 0.9273$$
$$\approx 0.6435$$

In one revolution, the maximum occurs at approximately 0.6435.

Where the minimum value occurs is found in a similar manner.

Exercises 9.4

In Exercises 1–27, find all solutions of the equation in the interval $[0, 2\pi]$.

1. $\sin^2 x + 3\cos^2 x = 0$
2. $\sin 2x + \cos x = 0$
3. $\cos 2x - \sin x = 1$
4. $\sin\frac{x}{2} = 1 - \cos x$
5. $4\sin^2\left(\frac{x}{2}\right) + \cos^2 x = 2$
6. $\sin 4x - \sin 2x = 0$
7. $\sin x \sin\frac{1}{2}x = 1 - \cos x$
8. $\sin 2t \cos t - \cos 2t \sin t = 0$
9. $\sin 2x \sin x + \cos x = 0$
10. $\cos x = \cos\frac{1}{2}x$ (Check for extraneous solutions.)
11. $\sin 2x + \cos 2x = 0$
12. $\cos 4x \cos x - \sin 4x \sin x = 0$
13. $\sin 4x = \cos 2x$
14. $\cos 2x + \sin^2 x = 0$
15. $2\cos^2 x - 2\cos 2x = 1$
16. $\cos\left(x + \frac{\pi}{2}\right) - \sin x = 1$
17. $\sin\left(x + \frac{\pi}{2}\right) + \cos x = 1$
18. $\sin x - \sqrt{3}\cos x = 0$
19. $\sin x + \cos x = 0$
20. $\sin 2x - \cos x = 0$
21. $\cos 2x + \cos x = 0$
22. $(\sin x - \cos x)^2 = 1$

15. $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
16. $x = \frac{7\pi}{6}, \frac{11\pi}{6}$
17. $x = \frac{\pi}{3}, \frac{5\pi}{3}$
18. $x = \frac{\pi}{3}, \frac{4\pi}{3}$
19. $x = \frac{3\pi}{4}, \frac{7\pi}{4}$
20. $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$
21. $x = \frac{\pi}{3}, \frac{5\pi}{3}, \pi$
22. $x = 0, \pi, 2\pi, \frac{\pi}{2}, \frac{3\pi}{2}$

23. $\sin x \cos x + \frac{1}{2} = 0$

24. $\sin^2\left(\frac{x}{2}\right) + \cos x = 0$

25. $\csc^2 \frac{x}{2} = 2 \sec x$

26. $-\sqrt{2} \sin x + \sqrt{2} \cos x = 1$

27. $2 \sin x - 2 \cos x = \sqrt{2}$

In Exercises 28–31, find the solution to each equation in the interval $[-\pi, \pi]$.

28. $\sin x + \cos x = 1$

29. $\sin^2 x + \cos 2x = 1$

30. $\sin 2x + \cos x = 0$

31. $-\cos x + \sqrt{3} \sin x = 1$

In Exercises 32–35, solve each equation in $[0, 2\pi)$.

32. $\sin 4x - \sin 2x = \sin x$

33. $\sin 2x = \sin \frac{1}{2} x$

34. $\cos 3x + \cos x = 0$

35. $\sin 3x - \sin x = 0$

In Exercises 36–40,

a. Express each function in the form $f(x) = k \sin(x + \alpha)$.
b. Find the maximum value that $f(x)$ can assume.
c. Find all values of x in $[0, 2\pi]$ that give the maximum value of $f(x)$.

36. $f(x) = \sqrt{3} \sin x - \cos x$

37. $f(x) = \sin x + \sqrt{3} \cos x$

38. $f(x) = 2 \sin x + 2 \cos x$

39. $f(x) = \sin x - \cos x$

40. $f(x) = 4 \sin x - 3 \cos x$

23. $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

24. $x = \pi$

25. $x = \frac{\pi}{3}, \frac{5\pi}{3}$

26. $x = \frac{\pi}{12}, \frac{17\pi}{12}$

27. $x = \frac{5\pi}{12}, \frac{13\pi}{12}$

28. $x = 0, \frac{\pi}{2}$

29. $x = -\pi, 0, \pi$

30. $x = -\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{6}, -\frac{5\pi}{6}$

31. $x = \frac{\pi}{3}, -\pi, \pi$

32. $x = 0, \pi, 2\pi, \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}$

33. $x = \frac{2\pi}{5}, \frac{6\pi}{5}, 2\pi, 0, \frac{4\pi}{3}$

34. $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

35. $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, 0, \pi, 2\pi$

36. **a.** $f(x) = 2 \sin\left(x + \frac{11\pi}{6}\right)$
b. 2
c. $x = \frac{2\pi}{3}$

37. **a.** $f(x) = 2 \sin\left(x + \frac{\pi}{3}\right)$
b. 2
c. $x = \frac{\pi}{6}$

38. **a.** $f(x) = 2\sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$
b. $2\sqrt{2}$
c. $x = \frac{\pi}{4}$

39. **a.** $f(x) = \sqrt{2} \sin\left(x + \frac{7\pi}{4}\right)$
b. $\sqrt{2}$
c. $x = \frac{3\pi}{4}$

40. **a.** $f(x) = 5 \sin(x - 0.6435)$
b. 5
c. $x \approx 2.2143$