45. $\dfrac{1}{\csc x - \sin x} = \sec x \tan x$

46. $\dfrac{1 + \csc x}{\csc x} = \dfrac{\cos^2 x}{1 - \sin x}$

47. $\dfrac{\sin x - \cos x}{\tan x} = \dfrac{\tan x}{\sin x + \cos x}$

48. $\dfrac{\cot x}{\csc x - 1} = \dfrac{\csc x + 1}{\cot x}$

In Exercises 49–52, half of an identity is given. Graph this half in a viewing window with $-2\pi \le x \le 2\pi$ and write a conjecture as to what the right side of the identity is. Then prove your conjecture.

49. $1 - \dfrac{\sin^2 x}{1 + \cos x} = ?$ *Hint:* What familiar function has a graph that looks like this?

50. $\dfrac{1 + \cos x - \cos^2 x}{\sin x} - \cot x = ?$

51. $(\sin x + \cos x)(\sec x + \csc x) - \cot x - 2 = ?$

52. $\cos^3 x(1 - \tan^4 x + \sec^4 x) = ?$

In Exercises 53–66, prove the identity.

53. $\dfrac{1 - \sin x}{\sec x} = \dfrac{\cos^3 x}{1 + \sin x}$

54. $\dfrac{\sin x}{1 - \cot x} = \dfrac{\cos x}{1 - \tan x} = \cos x + \sin x$

55. $\dfrac{\cos x}{1 - \sin x} = \sec x + \tan x$

56. $\dfrac{1 + \sec x}{\tan x + \sin x} = \csc x$

57. $\dfrac{\cos x \cot x}{\cot x - \cos x} = \dfrac{\cot x + \cos x}{\cos x \cot x}$

58. $\dfrac{\cos^3 x - \sin^3 x}{\cos x - \sin x} = 1 + \sin x \cos x$

59. $\log_{10}(\cot x) = -\log_{10}(\tan x)$

60. $\log_{10}(\sec x) = -\log_{10}(\cos x)$

61. $\log_{10}(\csc x + \cot x) = -\log_{10}(\csc x - \cot x)$

62. $\log_{10}(\sec x + \tan x) = -\log_{10}(\sec x - \tan x)$

63. $\tan x - \tan y = -\tan x \tan y(\cot x - \cot y)$

64. $\dfrac{\tan x - \tan y}{\cot x - \cot y} = -\tan x \tan y$

65. $\dfrac{\cos x - \sin y}{\cos y - \sin x} = \dfrac{\cos y + \sin x}{\cos x + \sin y}$

66. $\dfrac{\tan x + \tan y}{\cot x + \cot y} = \dfrac{\tan x \tan y - 1}{1 - \cot x \cot y}$

9.2 Addition and Subtraction Identities

Objectives

- Use the addition and subtraction identities for sine, cosine, and tangent functions
- Use the cofunction identities

Many times, the input, or *argument*, of the sine or cosine function is the sum or difference of two angles, and you may need to simplify the expression. Be careful not to make this common student error.

$$\sin\left(x + \frac{\pi}{6}\right) \text{ is not } \sin x + \sin\frac{\pi}{6}$$

Graphing Exploration

Verify graphically that the expressions above do NOT form an identity by graphing $Y_1 = \sin\left(x + \frac{\pi}{6}\right)$ and $Y_2 = \sin x + \sin\frac{\pi}{6}$.

21. $\dfrac{\sin(-x)}{\cos(-x)} = \dfrac{-\sin x}{\cos x} = -\tan x$

22. not an identity

23. $\cot(-x) = \dfrac{\cos(-x)}{\sin(-x)}$
$= \dfrac{\cos x}{-\sin x} = -\cot x$

24. $\sec(-x) = \dfrac{1}{\cos(-x)}$
$= \dfrac{1}{\cos x} = \sec x$

25. not an identity

26. $\sec^4 x - \tan^4 x$
$= (\sec^2 x - \tan^2 x)(\sec^2 x + \tan^2 x)$
$= 1(\sec^2 x + \tan^2 x)$
$= ((1 + \tan^2 x) + \tan^2 x)$
$= 1 + 2\tan^2 x$

27. $\sec^2 x - \csc^2 x$
$= (1 + \tan^2 x) - (1 + \cot^2 x)$
$= \tan^2 x - \cot^2 x$

28. $\sec^2 x + \csc^2 x = \dfrac{1}{\cos^2 x} + \dfrac{1}{\sin^2 x}$
$= \dfrac{\sin^2 x + \cos^2 x}{\cos^2 x \sin^2 x} = \dfrac{1}{\sin^2 x \cos^2 x}$
$= \left(\dfrac{1}{\sin^2 x}\right)\left(\dfrac{1}{\cos^2 x}\right) = \csc^2 x \sec^2 x$

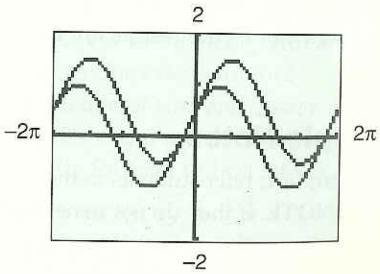
29–66. *See p.* 1070–1072.

Section 9.2 Addition and Subtraction Identities

Teaching Notes

Students may think $\sin(x + y) = \sin x + \sin y$. This is not true, as shown in the Graphing Exploration.

Solution to the **Graphing Exploration:**



The graphs are different, so $\sin\left(x + \frac{\pi}{6}\right) = \sin x + \sin\frac{\pi}{6}$ is NOT an identity.

in^2x
ot^2x

The exploration shows that $\sin(x + y) \neq \sin x + \sin y$ because it is fa
when $y = \frac{\pi}{6}$. So, is there an identity involving $\sin(x + y)$?

Graphing Exploration

Graph $Y_1 = \sin\left(x + \frac{\pi}{6}\right)$ and $Y_2 = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$ on the same screen. Do the graphs appear identical?

The exploration suggests that $\sin\left(x + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$ may b an identity. Furthermore, note that the coefficients on the right side ca be expressed in terms of $\frac{\pi}{6}$: $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\sin\frac{\pi}{6} = \frac{1}{2}$. In other words the following equation appears to be an identity.

$$\sin\left(x + \frac{\pi}{6}\right) = \sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6}$$

Graphing Exploration

Graph $Y_1 = \sin(x + 5)$ and $Y_2 = \sin x \cos 5 + \cos x \sin 5$ on the same screen. Do the graphs appear identical? What identity does this suggest? Repeat the process with some other number in place of 5. Are the results the same?

The equations examined in the discussion and exploration above ar examples of the first identity listed below. Each identity can be confirmed by assigning a constant value to y and then graphing each side of th equation, as in the Graphing Exploration above.

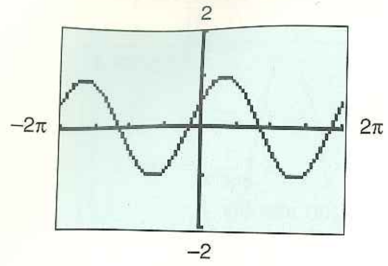
Addition and Subtraction Identities for Sine and Cosine

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

The addition and subtraction identities are important trigonometric identities. You should become familiar with the examples and special cases that follow.

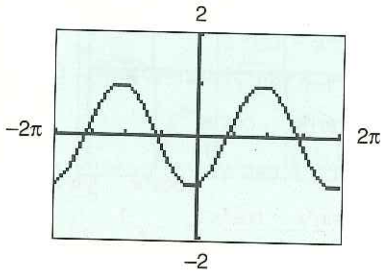
Teaching Notes

Solution to the first Graphing Exploration:



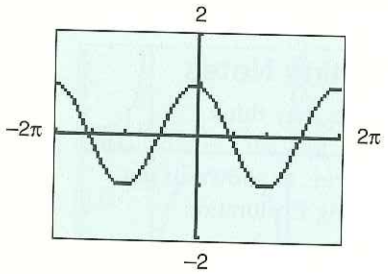
Yes, the graphs appear identical.

Solution to the second Graphing Exploration:



Yes, the graphs appear identical.

This suggests the identity $\sin(x + 5) = \sin x \cos 5 + \cos x \sin 5$. To repeat the process, students must use another number in place of 5. The graphs of $Y_1 = \sin(x + 8)$ and $Y_2 = \sin x \cos 8 + \cos x \sin 8$ are shown below (note that 8 is just one possible number that can be used).



Again, the graphs of Y_1 and Y_2 appear to be identical, so the results are the same.

Example Notes

For Example 1, refer students to the margin NOTE. If they do not remember all the necessary exact function values, refer them to pages 448–449, or the inside of the back cover.

After completing Example 2, point out that the result is the identity $\sin(\pi - y) = \sin y$ that was introduced on page 459.

NOTE In order to use addition or subtraction identities to find exact values, first write the argument as a sum or difference of two terms for which exact values are known, such as $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, and π.

Example 1 Addition Identities

Use the addition identities to find the *exact* values of $\sin \frac{5\pi}{12}$ and $\cos \frac{5\pi}{12}$.

Solution

Because $\frac{5\pi}{12} = \frac{2\pi}{12} + \frac{3\pi}{12} = \frac{\pi}{6} + \frac{\pi}{4}$, apply the addition identities with $x = \frac{\pi}{6}$ and $y = \frac{\pi}{4}$.

$$\sin \frac{5\pi}{12} = \sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \sin \frac{\pi}{6} \cos \frac{\pi}{4} + \cos \frac{\pi}{6} \sin \frac{\pi}{4}$$
$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}(1 + \sqrt{3})}{4}$$

$$\cos \frac{5\pi}{12} = \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \cos \frac{\pi}{6} \cos \frac{\pi}{4} - \sin \frac{\pi}{6} \sin \frac{\pi}{4}$$
$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}(\sqrt{3} - 1)}{4}$$

Example 2 Subtraction Identity

Find $\sin(\pi - y)$.

Solution

Apply the subtraction identity for the sine function with $x = \pi$.

$$\sin(\pi - y) = \sin \pi \cos y - \cos \pi \sin y = 0 \cos y - (-1)\sin y = \sin y$$

Example 3 Addition Identity

Prove that $\cos x \cos y = \frac{1}{2}[\cos(x + y) + \cos(x - y)]$.

Solution

Begin with the more complicated right side and use the addition and subtraction identities for cosine to transform it into the left side.

$$\frac{1}{2}[\cos(x + y) + \cos(x - y)]$$
$$= \frac{1}{2}[(\cos x \cos y - \sin x \sin y) + (\cos x \cos y + \sin x \sin y)]$$
$$= \frac{1}{2}(\cos x \cos y + \cos x \cos y) = \frac{1}{2}(2 \cos x \cos y)$$
$$= \cos x \cos y$$

ADDITIONAL EXAMPLES

Example 1

Use the addition identities to find the exact values of $\sin \frac{17\pi}{12}$ and $\cos \frac{17\pi}{12}$.

$$\sin \frac{17\pi}{12} = \sin\left(\frac{2\pi}{3} + \frac{3\pi}{4}\right)$$
$$= \sin \frac{2\pi}{3} \cos \frac{3\pi}{4} + \cos \frac{2\pi}{3} \sin \frac{3\pi}{4}$$
$$= \frac{\sqrt{3}}{2} \cdot \left(-\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right) \cdot \frac{\sqrt{2}}{2}$$
$$= -\frac{\sqrt{2}(\sqrt{3} + 1)}{4}$$

$$\cos \frac{17\pi}{12} = \cos\left(\frac{2\pi}{3} + \frac{3\pi}{4}\right)$$
$$= \cos \frac{2\pi}{3} \cos \frac{3\pi}{4} - \sin \frac{2\pi}{3} \sin \frac{3\pi}{4}$$
$$= \left(-\frac{1}{2}\right) \cdot \left(-\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{2}(1 - \sqrt{3})}{4}$$

Example 2

Find $\cos\left(x - \frac{\pi}{2}\right)$.

$$\cos\left(x - \frac{\pi}{2}\right)$$
$$= \cos x \cos \frac{\pi}{2} + \sin x \sin \frac{\pi}{2}$$
$$= (\cos x)(0) + (\sin x)(1) = \sin x$$

Example 3

Prove that $2 \cos x \sin y = \sin(x + y) - \sin(x - y)$.

$$\sin(x + y) - \sin(x - y)$$
$$= (\sin x \cos y + \cos x \sin y) - (\sin x \cos y - \cos x \sin y)$$
$$= \cos x \sin y + \cos x \sin y$$
$$= 2 \cos x \sin y$$

Real-World Application

A surveyor in the field, with a nonscientific calculator and knowledge of sine and cosine of 45° and 30°, could use the Subtraction Identity for Sine to find sin 15°.

$$\sin 45° = \cos 45° = \frac{\sqrt{2}}{2} \approx 0.7071$$
$$\cos 30° = \frac{\sqrt{3}}{2} \approx 0.8660 \text{ and}$$
$$\sin 30° = \frac{1}{2} = 0.5$$
$$\sin 15° = \sin(45° - 30°)$$
$$= \sin 45° \cos 30° - \cos 45° \sin 30°$$
$$\approx 0.7071(0.8660) - 0.7071(0.5)$$
$$\approx 0.2588$$

COMMON ERROR ALERT

In **Example 4,** some students might try to reduce the fraction $\frac{\sin h}{h}$ to sin. Remind them that $\sin h$ represents a function value and that sin by itself in an expression is meaningless.

Example Notes

The work for the last step in **Example 5** is shown below. **Additional Example 5** will require similar work for the last step.

$$\frac{3\sqrt{7} + 14\sqrt{2}}{7 - 6\sqrt{14}} \cdot \frac{7 + 6\sqrt{14}}{7 + 6\sqrt{14}}$$
$$= \frac{21\sqrt{7} + 18\sqrt{7}\sqrt{14} + 98\sqrt{2} + 84\sqrt{2}\sqrt{14}}{49 - 36 \cdot 14}$$
$$= \frac{21\sqrt{7} + 18 \cdot 7\sqrt{2} + 98\sqrt{2} + 84 \cdot 2\sqrt{7}}{-455}$$
$$= \frac{21\sqrt{7} + 126\sqrt{2} + 98\sqrt{2} + 168\sqrt{7}}{-455}$$
$$= \frac{189\sqrt{7} + 224\sqrt{2}}{-455}$$
$$= \frac{7(27\sqrt{7} + 32\sqrt{2})}{-7(65)}$$
$$= -\frac{27\sqrt{7} + 32\sqrt{2}}{65}$$

Additional Example 4 and Exercise 35 are identical.

ADDITIONAL EXAMPLES

Example 4

Show that for the function $f(x) = \cos x$ and any number $h \neq 0$,

$$\frac{f(x + h) - f(x)}{h} = \cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right).$$

$$\frac{f(x + h) - f(x)}{h} = \frac{\cos(x + h) - \cos x}{h}$$
$$= \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$
$$= \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$
$$= \cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)$$

NOTE Recall that the difference quotient of a function f is

$$\frac{f(x + h) - f(x)}{h}.$$

Simplifying the Difference Quotient of a Trigonometric Function

The difference quotient is very important in calculus, and the addition identities are needed to simplify difference quotients of trigonometric functions.

Example 4 The Difference Quotient of $f(x) = \sin x$

Show that for the function $f(x) = \sin x$ and any number $h \neq 0$,

$$\frac{f(x + h) - f(x)}{h} = \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right).$$

Solution

Use the addition identity for $\sin(x + y)$ with $y = h$.

$$\frac{f(x + h) - f(x)}{h} = \frac{\sin(x + h) - \sin x}{h}$$
$$= \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$
$$= \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$
$$= \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$$

Addition and Subtraction Identities for the Tangent Function

The addition and subtraction identities for sine and cosine can be used to obtain the addition and subtraction identities for the tangent function.

Addition and Subtraction Identities for Tangent

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

A proof of these identities is outlined in Exercise 36.

Example 5 Addition and Subtraction Identities for Tangent

Find the exact values of $\sin(x + y)$ and $\tan(x + y)$ if x and y are numbers such that $0 < x < \frac{\pi}{2}$, $\pi < y < \frac{3\pi}{2}$, $\sin x = \frac{3}{4}$, and $\cos y = -\frac{1}{3}$. Deter-

ction
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function

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e number
$\frac{1}{3}$. Dete

mine in which of the following intervals $x + y$ lies: $\left(0, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \pi\right)$, $\left(\pi, \frac{3\pi}{2}\right)$, or $\left(\frac{3\pi}{2}, 2\pi\right)$.

NOTE See Figure 6.4-7 for the signs of the functions in each quadrant.

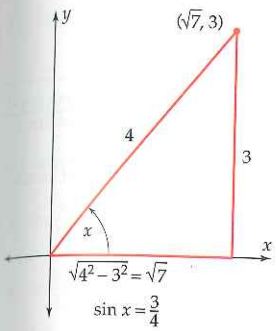


Figure 9.2-1a

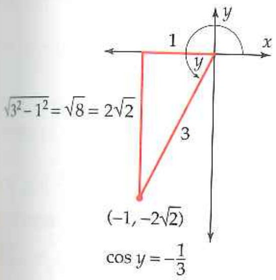


Figure 9.2-1b

Solution

Use the Pythagorean identity and the fact that cos x and tan x are positive in the first quadrant to obtain the following. See Figure 9.2-1a.

$$\cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \left(\frac{3}{4}\right)^2} = \sqrt{1 - \frac{9}{16}} = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{3}{4}}{\frac{\sqrt{7}}{4}} = \frac{3}{\sqrt{7}} = \frac{3\sqrt{7}}{7}$$

Because y lies between π and $\frac{3\pi}{2}$, its sine is negative. See Figure 9.2-1b.

$$\sin y = -\sqrt{1 - \cos^2 y} = -\sqrt{1 - \left(-\frac{1}{3}\right)^2} = -\sqrt{\frac{8}{9}} = -\frac{\sqrt{8}}{3} = -\frac{2\sqrt{2}}{3}$$

$$\tan y = \frac{\sin y}{\cos y} = \frac{\frac{-2\sqrt{2}}{3}}{\frac{-1}{3}} = \frac{-2\sqrt{2}}{3} \cdot \frac{3}{-1} = 2\sqrt{2}$$

The addition identities for sine and tangent show exact values.

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$= \frac{3}{4} \cdot \frac{-1}{3} + \frac{\sqrt{7}}{4} \cdot \frac{-2\sqrt{2}}{3} = \frac{-3}{12} - \frac{2\sqrt{14}}{12} = \frac{-3 - 2\sqrt{14}}{12}$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$= \frac{\frac{3\sqrt{7}}{7} + 2\sqrt{2}}{1 - \left(\frac{3\sqrt{7}}{7}\right)(2\sqrt{2})} = \frac{\frac{3\sqrt{7} + 14\sqrt{2}}{7}}{\frac{7 - 6\sqrt{14}}{7}}$$
$$= \frac{3\sqrt{7} + 14\sqrt{2}}{7 - 6\sqrt{14}} = -\frac{27\sqrt{7} + 32\sqrt{2}}{65}$$

Both the sine and tangent of $x + y$ are negative numbers. Therefore $x + y$ must be in the interval $\left(\frac{3\pi}{2}, 2\pi\right)$ because it is the only one of the four intervals in which both sine and tangent are negative.

Cofunction Identities

Special cases of the addition and subtraction identities are the cofunction identities.

ADDITIONAL EXAMPL

Example 5

Find the exact values of $\cos(x + y)$ and $\tan(x + y)$ if x and y are numbers such that $\frac{\pi}{2} < x < \pi$, $0 < y < \frac{\pi}{2}$, $\cos x =$ and $\sin y = \frac{1}{4}$. Determine in whic the following intervals $x + y$ lies: $\left(0, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \pi\right)$, $\left(\pi, \frac{3\pi}{2}\right)$, or $\left(\frac{3\pi}{2}, 2\right.$

$$\sin x = \sqrt{1 - \cos^2 x}$$
$$= \sqrt{1 - \left(-\frac{2}{3}\right)^2} = \sqrt{1 - \frac{4}{9}}$$
$$= \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{\sqrt{5}}{3}}{-\frac{2}{3}} = -\frac{\sqrt{5}}{2}$$

$$\cos y = \sqrt{1 - \sin^2 y}$$
$$= \sqrt{1 - \left(\frac{1}{4}\right)^2} = \sqrt{1 - \frac{1}{16}}$$
$$= \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4}$$

$$\tan y = \frac{\sin y}{\cos y} = \frac{\frac{1}{4}}{\frac{\sqrt{15}}{4}}$$
$$= \frac{1}{\sqrt{15}} = \frac{\sqrt{15}}{15}$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin$$
$$= -\frac{2}{3} \cdot \frac{\sqrt{15}}{4} - \frac{\sqrt{5}}{3} \cdot \frac{1}{4}$$
$$= \frac{-2\sqrt{15} - \sqrt{5}}{12}$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$= \frac{-\frac{\sqrt{5}}{2} + \frac{\sqrt{15}}{15}}{1 - \left(-\frac{\sqrt{5}}{2}\right)\left(\frac{\sqrt{15}}{15}\right)}$$
$$= \frac{\frac{-15\sqrt{5} + 2\sqrt{15}}{30}}{\frac{30 + \sqrt{75}}{30}} = \frac{-15\sqrt{5} + 2\sqrt{}}{30 + \sqrt{75}}$$
$$= \frac{-15\sqrt{5} + 2\sqrt{15}}{30 + 5\sqrt{3}}$$
$$= \frac{-15\sqrt{5} + 2\sqrt{15}}{30 + 5\sqrt{3}} \cdot \frac{30 - 5\sqrt{3}}{30 - 5\sqrt{3}}$$
$$= \frac{9\sqrt{15} - 32\sqrt{5}}{55}$$

Both $\cos(x + y)$ and $\tan(x + y)$ are negative, so $x + y$ is in the interval $\left(\frac{\pi}{2}, \pi\right)$.

Cofunction Identities

$$\sin x = \cos\left(\frac{\pi}{2} - x\right) \qquad \cos x = \sin\left(\frac{\pi}{2} - x\right)$$

$$\tan x = \cot\left(\frac{\pi}{2} - x\right) \qquad \cot x = \tan\left(\frac{\pi}{2} - x\right)$$

$$\sec x = \csc\left(\frac{\pi}{2} - x\right) \qquad \csc x = \sec\left(\frac{\pi}{2} - x\right)$$

The first cofunction identity is proved by using the identity for $\cos(x - y)$ with $\frac{\pi}{2}$ in place of x and x in place of y.

$$\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x = 0 \cdot \cos x + 1 \cdot \sin x = \sin x$$

Because the first cofunction identity is valid for *every* number x, it is also valid with the number $\frac{\pi}{2} - x$ in place of x.

$$\sin\left(\frac{\pi}{2} - x\right) = \cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right] = \cos x$$

Thus, the second cofunction identity is proved. The others now follow from these previous two. For instance,

$$\tan\left(\frac{\pi}{2} - x\right) = \frac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)} = \frac{\cos x}{\sin x} = \cot x$$

Also,

$$\csc\left(\frac{\pi}{2} - x\right) = \frac{1}{\sin\left(\frac{\pi}{2} - x\right)} = \frac{1}{\cos x} = \sec x$$

Example 6 Cofunction Identities

Verify that $\dfrac{\cos\left(x - \frac{\pi}{2}\right)}{\cos x} = \tan x$.

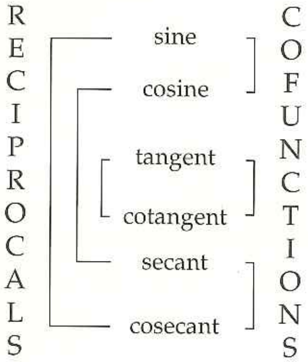
Solution

Beginning with the left side, the term $\cos\left(x - \frac{\pi}{2}\right)$ looks almost, but not quite, like the term $\cos\left(\frac{\pi}{2} - x\right)$ in the cofunction identity. But note that $-\left(x - \frac{\pi}{2}\right) = \frac{\pi}{2} - x$. Therefore,

Teaching Notes

Point out that:
Sine and cosine are cofunctions.
Tangent and cotangent are cofunctions.
Secant and cosecant are cofunctions.

The diagram below is a convenient way to remember the Reciprocal Identities (page 455) and the Cofunction Identities together:

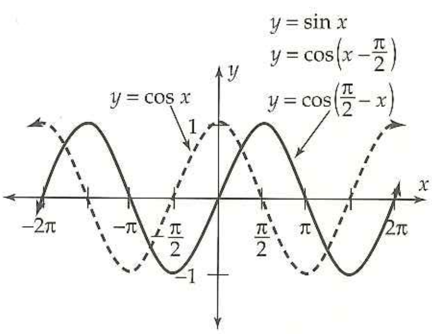


The following explanation of the Cofunction Identity $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ might be helpful to some students:

Using the Negative Angle Identity,

$$\cos\left(\frac{\pi}{2} - x\right) = \cos\left[-\left(\frac{\pi}{2} - x\right)\right] = \cos\left(x - \frac{\pi}{2}\right).$$

The graph of $y = \cos\left(x - \frac{\pi}{2}\right)$ is the graph of $y = \cos x$, shifted to the right $\frac{\pi}{2}$ units. Therefore, the graph of $y = \cos\left(\frac{\pi}{2} - x\right)$ is the graph of $y = \cos x$, shifted to the right $\frac{\pi}{2}$ units. Furthermore, it is easily verified by a table of values that the graph of $y = \sin x$ is the graph of $y = \cos x$, shifted to the right $\frac{\pi}{2}$ units. Therefore, the graphs of $y = \sin x$ and $y = \cos\left(\frac{\pi}{2} - x\right)$ are the same, as shown.



$$\frac{\cos\left(x - \frac{\pi}{2}\right)}{\cos x} = \frac{\cos\left[-\left(x - \frac{\pi}{2}\right)\right]}{\cos x} \quad \textit{negative angle identity with } x - \tfrac{\pi}{2} \textit{ in place of } x$$

$$= \frac{\cos\left(\frac{\pi}{2} - x\right)}{\cos x}$$

$$= \frac{\sin x}{\cos x} \quad \textit{cofunction identity}$$

$$= \tan x \quad \textit{quotient identity}$$

Exercises 9.2

In Exercises 1–12, find the exact value.

1. $\sin\frac{\pi}{12}$ $\frac{\sqrt{6}-\sqrt{2}}{4}$
2. $\cos\frac{\pi}{12}$ $\frac{\sqrt{2}+\sqrt{6}}{4}$
3. $\tan\frac{\pi}{12}$ $2-\sqrt{3}$
4. $\sin\frac{5\pi}{12}$ $\frac{\sqrt{6}+\sqrt{2}}{4}$
5. $\cot\frac{5\pi}{12}$ $2-\sqrt{3}$
6. $\cos\frac{7\pi}{12}$ $\frac{\sqrt{2}-\sqrt{6}}{4}$
7. $\tan\frac{7\pi}{12}$ $-2-\sqrt{3}$
8. $\cos\frac{11\pi}{12}$ $\frac{-\sqrt{2}-\sqrt{6}}{4}$
9. $\cot\frac{11\pi}{12}$ $-2-\sqrt{3}$
10. $\sin 75°$ *Hint:* $75° = 45° + 30°$. $\frac{\sqrt{6}+\sqrt{2}}{4}$
11. $\sin 105°$ $\frac{\sqrt{6}+\sqrt{2}}{4}$
12. $\cos 165°$ $\frac{-\sqrt{2}-\sqrt{6}}{4}$

In Exercises 13–18, rewrite the given expression in terms of $\sin x$ and $\cos x$.

13. $\sin\left(\frac{\pi}{2} + x\right)$ $\cos x$
14. $\cos\left(x + \frac{\pi}{2}\right)$ $-\sin x$
15. $\cos\left(x - \frac{3\pi}{2}\right)$ $-\sin x$
16. $\csc\left(x + \frac{\pi}{2}\right)$ $\frac{1}{\cos x}$
17. $\sec(x - \pi)$ $-\frac{1}{\cos x}$
18. $\cot(x + \pi)$ $\frac{\cos x}{\sin x}$

In Exercises 19–24, simplify the given expression.

19. $\sin 3 \cos 5 - \cos 3 \sin 5$ $-\sin 2$
20. $\sin 37° \sin 53° - \cos 37° \cos 53°$ 0
21. $\cos(x + y)\cos y + \sin(x + y)\sin y$ $\cos x$
22. $\sin(x - y)\cos y + \cos(x - y)\sin y$ $\sin x$
23. $\cos(x + y) - \cos(x - y)$ $-2\sin x \sin y$
24. $\sin(x + y) - \sin(x - y)$ $2\sin y \cos x$

25. If $\sin x = \frac{1}{3}$ and $0 < x < \frac{\pi}{2}$, then $\sin\left(\frac{\pi}{4} + x\right) = ?$ $\frac{4+\sqrt{2}}{6}$
26. If $\cos x = -\frac{1}{4}$ and $\frac{\pi}{2} < x < \pi$, then $\cos\left(\frac{\pi}{6} - x\right) = ?$ $\frac{-\sqrt{3}+\sqrt{15}}{8}$
27. If $\cos x = -\frac{1}{5}$ and $\pi < x < \frac{3\pi}{2}$, then $\sin\left(\frac{\pi}{3} - x\right) = ?$ $\frac{2\sqrt{6}-\sqrt{3}}{10}$
28. If $\sin x = -\frac{3}{4}$ and $\frac{3\pi}{2} < x < 2\pi$, then $\cos\left(\frac{\pi}{4} + x\right) = ?$ $\frac{\sqrt{14}+3\sqrt{2}}{8}$

In Exercises 29–34, assume that $\sin x = 0.8$ and $\sin y = \sqrt{0.75}$ and that x and y lie between 0 and $\frac{\pi}{2}$. Evaluate the given expressions.

29. $\cos(x + y)$ -0.393
30. $\sin(x + y)$ 0.9196
31. $\cos(x - y)$ 0.993
32. $\sin(x - y)$ -0.1196
33. $\tan(x + y)$ -2.34
34. $\tan(x - y)$ -0.1204

35. If $f(x) = \cos x$ and h is a fixed nonzero number, prove that the difference quotient is
$$\frac{f(x+h) - f(x)}{h} = \cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right).$$

36. Prove the addition and subtraction identities for the tangent function. *Hint:*
$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)}$$
Use the addition identities on the numerator and denominator; then divide both numerator and denominator by $\cos x \cos y$, and simplify.

41. $\sin(u + v + w)$
$= \sin u \cos v \cos w + \cos u \sin v \cos w + \cos u \cos v \sin w - \sin u \sin v \sin w$

42. $\cos(x + y + z)$
$= \cos x \cos y \cos z - \sin x \sin y \cos z - \sin x \cos y \sin z - \cos x \sin y \sin z$

43. Since $y = \frac{\pi}{2} - x$,
$\sin y = \sin\left(\frac{\pi}{2} - x\right) = \cos x.$
Hence, $\sin^2 x + \sin^2 y = \sin^2 x + \cos^2 x = 1$

44. $$\cot(x+y) = \frac{\cos(x+y)}{\sin(x+y)} = \frac{\cos x\cos y - \sin x \sin y}{\sin x \cos y + \sin y \cos x} = \frac{\frac{\cos x \cos y - \sin x \sin y}{\sin x \sin y}}{\frac{\sin x \cos y + \sin y \cos x}{\sin x \sin y}} = \frac{\cot x \cot y - 1}{\cot x + \cot y}$$

ADDITIONAL EXAMPLES

Example 6

Verify that $\frac{\sin\left(x - \frac{\pi}{2}\right)}{\cos x} = -1$.

$$\frac{\sin\left(x - \frac{\pi}{2}\right)}{\cos x}$$
$$= \frac{\sin\left[-\left(\frac{\pi}{2} - x\right)\right]}{\cos x} \quad a - b = -(b - a)$$
$$= \frac{-\sin\left(\frac{\pi}{2} - x\right)}{\cos x} \quad \sin t = -\sin(-t)$$
$$= \frac{-\cos x}{\cos x} \quad \text{cofunction identity}$$
$$= -1$$

Exercises 9.2

ANSWERS

35. $$\frac{f(x+h) - f(x)}{h} = \frac{\cos(x+h) - \cos x}{h} = \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} = \frac{\cos x \cos h - \cos x}{h} - \frac{\sin x \sin h}{h} = \cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)$$

36. $$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \sin y \cos x}{\cos x \cos y - \sin x \sin y} = \frac{\frac{\sin x \cos y + \sin y \cos x}{\cos x \cos y}}{\frac{\cos x \cos y - \sin x \sin y}{\cos x \cos y}} = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

37. $\sin(x + y) = -\frac{44}{125}$; $\tan(x + y) = \frac{44}{117}$; $x + y$ is in the third quadrant

38. $\sin(x + y) = \frac{-3 - 2\sqrt{14}}{12}$; $\cos(x + y) = \frac{6\sqrt{2} - \sqrt{7}}{12}$; $\tan(x + y) = \frac{-3 - 2\sqrt{14}}{6\sqrt{2} - \sqrt{7}}$; $x + y$ is in the fourth quadrant

39. $\cos(x + y) = -\frac{56}{65}$; $\tan(x + y) = \frac{33}{56}$; $x + y$ is in the third quadrant

40. $\sin(x - y) = \frac{-4\sqrt{2} - \sqrt{5}}{9}$; $\tan(x - y) = \frac{-4\sqrt{2} - \sqrt{5}}{2 - 2\sqrt{10}}$; $x - y$ is in the third quadrant

37. If x is in the first quadrant and y is in the second quadrant, $\sin x = \frac{24}{25}$, and $\sin y = \frac{4}{5}$, find the exact value of $\sin(x + y)$ and $\tan(x + y)$ and the quadrant in which $x + y$ lies.

38. If x and y are in the second quadrant, $\sin x = \frac{1}{3}$, and $\cos y = -\frac{3}{4}$, find the exact value of $\sin(x + y)$, $\cos(x + y)$, and $\tan(x + y)$ and the quadrant in which $x + y$ lies.

39. If x is in the first quadrant and y is in the second quadrant, $\sin x = \frac{4}{5}$, and $\cos y = -\frac{12}{13}$, find the exact value of $\cos(x + y)$ and $\tan(x + y)$ and the quadrant in which $x + y$ lies.

40. If x is in the fourth quadrant and y is in the first quadrant, $\cos x = \frac{1}{3}$, and $\cos y = \frac{2}{3}$, find the exact value of $\sin(x - y)$ and $\tan(x - y)$ and the quadrant in which $x - y$ lies.

41. Express $\sin(u + v + w)$ in terms of sines and cosines of u, v, and w. *Hint:* First apply the addition identity with $x = u + v$ and $y = w$.

42. Express $\cos(x + y + z)$ in terms of sines and cosines of x, y, and z.

43. If $x + y = \frac{\pi}{2}$, show that $\sin^2 x + \sin^2 y = 1$.

44. Prove that $\cot(x + y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$.

In Exercises 45–56, prove the identity.

45. $\sin(x - \pi) = -\sin x$

46. $\cos(x - \pi) = -\cos x$

47. $\cos(\pi - x) = -\cos x$

48. $\tan(\pi - x) = -\tan x$

49. $\sin(x + \pi) = -\sin x$

50. $\cos(x + \pi) = -\cos x$

51. $\tan(x + \pi) = \tan x$

52. $\sin x \cos y = \frac{1}{2}[\sin(x + y) + \sin(x - y)]$

53. $\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$

54. $\cos x \sin y = \frac{1}{2}[\sin(x + y) - \sin(x - y)]$

55. $\cos(x + y)\cos(x - y) = \cos^2 x \cos^2 y - \sin^2 x \sin^2 y$

56. $\sin(x + y)\sin(x - y) = \sin^2 x \cos^2 y - \cos^2 x \sin^2 y$

In Exercises 57–66, determine graphically whether the equation could not possibly be an identity (by choosing a numerical value for y and graphing both sides), or write a proof that it is.

57. $\dfrac{\cos(x - y)}{\sin x \cos y} = \cot x + \tan y$

58. $\dfrac{\cos(x + y)}{\sin x \cos y} = \cot x - \tan y$

59. $\sin(x - y) = \sin x - \sin y$

60. $\cos(x + y) = \cos x + \cos y$

61. $\dfrac{\sin(x + y)}{\sin(x - y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$

62. $\dfrac{\sin(x + y)}{\sin(x - y)} = \dfrac{\cot y + \cot x}{\cot y - \cot x}$

63. $\dfrac{\cos(x + y)}{\cos(x - y)} = \dfrac{\cot x + \tan y}{\cot x - \tan y}$

64. $\dfrac{\cos(x - y)}{\cos(x + y)} = \dfrac{\cot y + \tan x}{\cot y - \tan x}$

65. $\tan(x + y) = \tan x + \tan y$

66. $\cot(x - y) = \cot x - \cot y$

45. $\sin(x - \pi)$
$= \sin x \cos \pi - \cos x \sin \pi$
$= (\sin x)(-1) - (\cos x)(0)$
$= -\sin x$

46. $\cos(x - \pi)$
$= \cos x \cos \pi + \sin x \sin \pi$
$= \cos x(-1) + \sin x(0) = -\cos x$

47. $\cos(\pi - x)$
$= \cos \pi \cos x + \sin \pi \sin x$
$= (-1)\cos x + (0)\sin x = -\cos x$

48. $\tan(\pi - x) = \dfrac{\sin(\pi - x)}{\cos(\pi - x)}$
$= \dfrac{\sin \pi \cos x - \sin x \cos \pi}{\cos \pi \cos x + \sin \pi \sin x}$
$= \dfrac{(0)\cos x - \sin x(-1)}{(-1)\cos x + (0)\sin x}$
$= -\tan x$

49. $\sin(x + \pi)$
$= \sin x \cos \pi + \cos x \sin \pi$
$= (\sin x)(-1) + (\cos x)(0)$
$= -\sin x$

50. $\cos(x + \pi)$
$= \cos x \cos \pi - \sin x \sin \pi$
$= \cos x(-1) - \sin x(0)$
$= -\cos x$

51. By Exercises 49 and 50,
$\tan(x + \pi) = \dfrac{\sin(x + \pi)}{\cos(x + \pi)}$
$= \dfrac{-\sin x}{-\cos x} = \tan x$

52. $\frac{1}{2}[\sin(x + y) + \sin(x - y)]$
$= \frac{1}{2}[(\sin x \cos y + \sin y \cos x) + (\sin x \cos y - \sin y \cos x)]$
$= \frac{1}{2}(2 \sin x \cos y)$
$= \sin x \cos y$

53. $\frac{1}{2}[\cos(x - y) - \cos(x + y)]$
$= \frac{1}{2}[(\cos x \cos y + \sin x \sin y) - (\cos x \cos y - \sin x \sin y)]$
$= \frac{1}{2}(2 \sin x \sin y) = \sin x \sin y$

54. $\frac{1}{2}[\sin(x + y) - \sin(x - y)]$
$= \frac{1}{2}[(\sin x \cos y + \sin y \cos x) - (\sin x \cos y - \sin y \cos x)]$
$= \frac{1}{2}(2 \sin y \cos x) = \cos x \sin y$

55. $\cos(x + y)\cos(x - y)$
$= (\cos x \cos y - \sin x \sin y) \cdot (\cos x \cos y + \sin x \sin y)$
$= (\cos x \cos y)^2 - (\sin x \sin y)^2$
$= \cos^2 x \cos^2 y - \sin^2 x \sin^2 y$

56. $\sin(x + y)\sin(x - y)$
$= (\sin x \cos y + \sin y \cos x) \cdot (\sin x \cos y - \sin y \cos x)$
$= \sin^2 x \cos^2 y - \sin^2 y \cos^2 x$

57. $\dfrac{\cos(x - y)}{\sin x \cos y} = \dfrac{\cos x \cos y + \sin x \sin y}{\sin x \cos y}$
$= \dfrac{\cos x}{\sin x} + \dfrac{\sin y}{\cos y}$
$= \cot x + \tan y$

58. $\dfrac{\cos(x + y)}{\sin x \cos y} = \dfrac{\cos x \cos y - \sin x \sin y}{\sin x \cos y}$
$= \dfrac{\cos x \cos y}{\sin x \cos y} - \dfrac{\sin x \sin y}{\sin x \cos y}$
$= \cot x - \tan y$

59–66. *See p. 1072.*