Chapter Outline

Interdependence of Sections

9.1 $\longrightarrow$ 9.2 $\longrightarrow$ 9.3 $\longrightarrow$ 9.4

The basic trigonometric identities, which were discussed in Chapter 6 and used in Chapter 8, are not the only identities that are useful in rewriting trigonometric expressions and in solving trigonometric equations. This chapter presents many widely used trigonometric identities and specific methods for solving particular forms of trigonometric equations.

equation sometimes have one solution

identity → left really does equal right side no matter what

9.1 Identities and Proofs

Objectives

- Identify possible identities by using graphs
- Apply strategies to prove identities

Recall that an identity is an equation that is true for all values of the variable for which every term of the equation is defined. Several trigonometric identities have been discussed in previous sections. This section will introduce other identities and discuss techniques used to verify that an equation is an identity.

Trigonometric identities can be used for simplifying expressions, rewriting the rules of trigonometric functions, and performing numerical calculations. There are no hard and fast rules for dealing with identities, but some suggestions follow. The phrases "prove the identity" and "verify the identity" mean "prove that the given equation is an identity."

Section 9.1 Identities and Proofs

Teaching Notes

In this chapter, students will be asked to prove (verify) identities. Point out how *proving* or *verifying* an identity is different from *solving* an equation.

To solve an equation, you *find the set of all numbers that make the equation true*. For example, students solved $\tan x \cos^2 x = \tan x$ in Section 8.3 (page 543). All solutions were given by $x = k\pi$, where k is any integer.

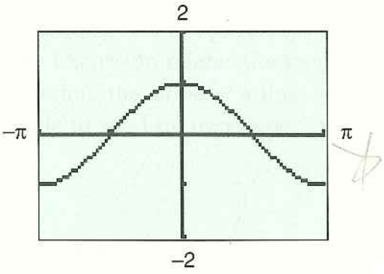
To prove an identity, you *show that an equation is true for all numbers* (for which all terms of the equation are defined). The equation $\tan x = \dfrac{\sin x}{\cos x}$ (from Section 6.4) is an identity because it is true for all values of x except values for which $\cos x = 0$.

In this chapter, students will be asked to prove new identities.

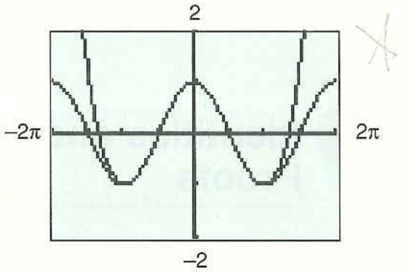
Teaching Notes

In Figures 9.1-1 and 9.1-2a, one graph is darker than another to emphasize that the graphs are different. Point out to students that when they do **Graphical Testing,** they should *not* make one graph darker than the other. Making one graph darker than the other might make it harder to detect whether the functions are different.

Solution to the **Graphing Exploration:**



Yes, the graphs of the functions with $-\pi \le x \le \pi$ appear identical.



No, the equation is not an identity; the graphs are not identical, as seen in the viewing window $-2\pi \le x \le 2\pi$.

Math Background

In the **Graphing Exploration,** the graphs appear to be the same in the window with $-\pi \le x \le \pi$. Recall (page 521 Exercise 5) that the series

$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ approximates $\cos x$.

Note that the series (page 521 Example 2)

$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots +$

$(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$ approximates $\sin x$.

it is true for some x-values but not all

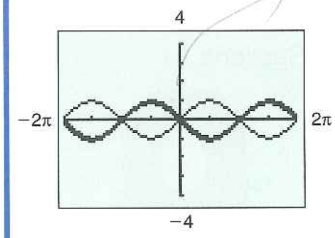


Figure 9.1-1

Graphical Testing

When presented with a trigonometric equation that *might* be an identity, it is a good idea to determine graphically whether or not this is possible. For instance, the equation $\cos\left(\frac{\pi}{2} + x\right) = \sin t$ can be tested to determine if it is possibly an identity by graphing $Y_1 = \cos\left(\frac{\pi}{2} + x\right)$ and $Y_2 = \sin x$ on the same screen, as shown in Figure 9.1-1 where the graph of Y_1 is darker than the graph of Y_2. Because the graphs are different, it can be concluded that the equation is not an identity.

Any equation can be tested by simultaneously graphing the two functions whose rules are given by the left and right sides of the equation. If the graphs are different, the equation is not an identity. If the graphs appear to be the same, then it is possible that the equation is an identity. However,

The fact that the graphs of both sides of an equation appear identical does not prove that the equation is an identity, as the following exploration demonstrates.

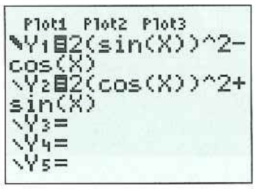
Calc might be off or extend window

Graphing Exploration

In the viewing window with $-\pi \le x \le \pi$ and $-2 \le y \le 2$, graph both sides of the equation

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40{,}320}$$

Do the graphs appear identical? Now change the viewing window so that $-2\pi \le x \le 2\pi$. Is the equation an identity?



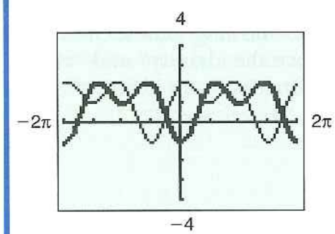


Figure 9.1-2a

Example 1 Graphical Identity Testing

Is either of the following equations an identity?

a. $2\sin^2 x - \cos x = 2\cos^2 x + \sin x$

b. $\dfrac{1 + \sin x - \sin^2 x}{\cos x} = \cos x + \tan x$

Solution

Test each equation graphically to see if it might be an identity by graphing each side of the equation.

a. Graph $Y_1 = 2\sin^2 x - \cos x$ and $Y_2 = 2\cos^2 x + \sin x$ on the same screen, as shown in Figure 9.1-2a. The graph of Y_1 is shown darker than Y_2. Because the graphs are *not* the same, the equation $2\sin^2 x - \cos x = 2\cos^2 x + \sin x$ is not an identity.

b. The graph shown in Figure 9.1-2b suggests that

$$\frac{1 + \sin x - \sin^2 x}{\cos x} = \cos x + \tan x$$

may be an identity, but the proof that it actually is an identity must be done algebraically.

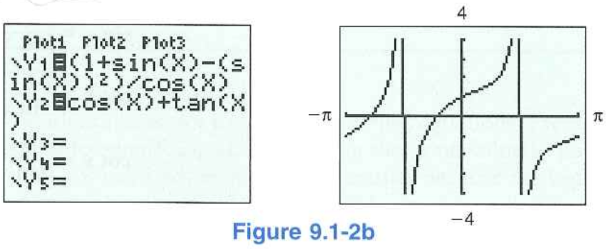


Figure 9.1-2b

CAUTION

Be sure to use parentheses correctly when entering each function to be graphed.

Example 2 Finding an Identity

Find an equation involving $2 \sin x \cos x$ that could possibly be an identity.

Solution

The graph of $y = 2 \sin x \cos x$ is shown in Figure 9.1-3a. Does it look familiar? At first it looks like the graph of $y = \sin x$, but there is an important difference. The function graphed in Figure 9.1-3a has a period of π. As was shown in Section 7.3, the graph of $y = \sin 2x$ looks like the sine graph but has a period of π.

The graphs $Y_1 = 2 \sin x \cos x$ and $Y_2 = \sin 2x$ are shown in Figures 9.1-3a and 9.1-3b. Because the graphs appear identical, $2 \sin x \cos x = \sin 2x$ may be an identity.

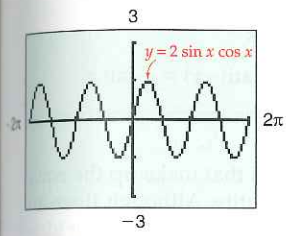


Figure 9.1-3a

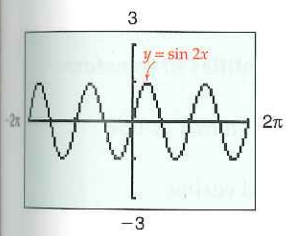


Figure 9.1-3b

Proving Identities

A useful feature of trigonometric functions is that they can be written in many ways. One form may be easier to use in one situation, and a different form of the same function may be more useful in another.

The elementary identities that were given in Section 6.5 are summarized for your reference on the following page. Memorizing these identities will benefit you greatly in the future.

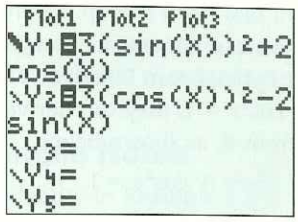
NOTE The definitions of the basic trigonometric ratios may help you remember the quotient and reciprocal identities. The shapes of the graphs of sine, cosine, and tangent may help you remember the periodicity and negative angle identities. Also, if you can remember the first of the Pythagorean identities, which is based on the Pythagorean Theorem, the other two can easily be derived from it.

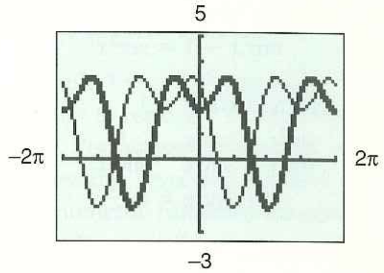
ADDITIONAL EXAMPLES

Example 1

Is either of the following equations an identity?

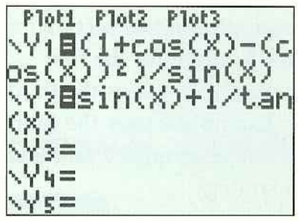
a. $3 \sin^2 x + 2 \cos x = 3 \cos^2 x - 2 \sin x$

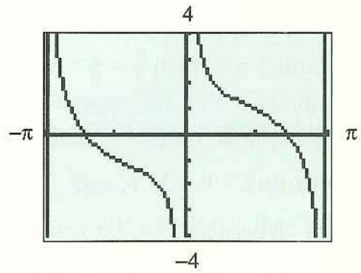




The graph of Y_1 is shown darker than Y_2. Because the graphs are not the same, the equation is not an identity.

b. $\dfrac{1 + \cos x - \cos^2 x}{\sin x} = \sin x + \cot x$





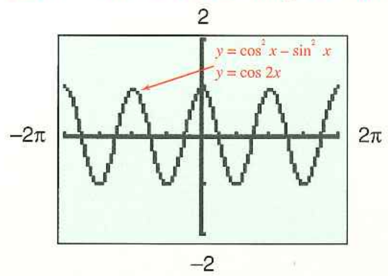
Because the graphs appear to be the same, the equation *may* be an identity.

Example 2

Find an equation involving $\cos^2 x - \sin^2 x$ that could possibly be an identity.

The graph of $\cos^2 x - \sin^2 x$ looks like the graph of $\cos x$, except it has a period of π.

The graphs of $Y_1 = \cos^2 x - \sin^2 x$ and $Y_2 = \cos 2x$ appear to be identical. Therefore, $\cos^2 x - \sin^2 = \cos 2x$ may be an identity.



Teaching Notes

Most of the **Basic Trigonometric Identities** on this page are from Section 6.5 (page 460). Emphasize the importance of memorizing them. They may also be found on the inside of the back cover.

If students can remember the first Pythagorean Identity, $\sin^2 x + \cos^2 x = 1$, they can derive the others from it, as shown below.

$$\sin^2 x + \cos^2 x = 1$$
$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$
$$\tan^2 x + 1 = \sec^2 x$$

$$\sin^2 x + \cos^2 x = 1$$
$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$
$$1 + \cot^2 x = \csc^2 x$$

If students remember the period of each trigonometric function, it will help them remember the Periodicity Identities. For example, since the period of $\sin x$ is 2π, $\sin(x \pm 2\pi) = \sin x$.

The **Strategies for Proving Trigonometric Identities** are used in the Examples in this section. **Examples 3** and **5** use the first strategy, **Example 4** uses the third strategy, **Example 6** uses the fourth strategy, and **Examples 7** and **8** use the fifth strategy.

Strategy 5 is based on the following property of proportions:
If $b \neq 0$ and $d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. Illustrate this property with numbers, such as: $\frac{2}{3} = \frac{6}{9}$ because $2 \cdot 9 = 3 \cdot 6$, and $2 \cdot 9 = 3 \cdot 6$ because $\frac{2}{3} = \frac{6}{9}$.

Basic Trigonometry Identities

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Reciprocal Identities

$$\sin x = \frac{1}{\csc x} \qquad \cos x = \frac{1}{\sec x}$$
$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x}$$
$$\cot x = \frac{1}{\tan x} \qquad \tan x = \frac{1}{\cot x}$$

Periodicity Identities

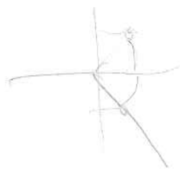
$$\sin(x \pm 2\pi) = \sin x \qquad \cos(x \pm 2\pi) = \cos x$$
$$\csc(x \pm 2\pi) = \csc x \qquad \sec(x \pm 2\pi) = \sec x$$
$$\tan(x \pm \pi) = \tan x \qquad \cot(x \pm \pi) = \cot x$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Negative Angle Identities

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$



Just looking at the graphs of the two expressions that make up the equation is not enough to guarantee that it is an identity. Although there are no exact rules for simplifying trigonometric expressions or proving identities, there are some common strategies that are often helpful.

+ Plans of Attack

Strategies for Proving Trigonometric Identities

1. **Use algebra and *previously* proven identities to transform one side of the equation into the other.**
2. **If possible, write the entire equation in terms of one trigonometric function.**
3. **Express everything in terms of sine and cosine.**
4. **Deal separately with each side of the equation $A = B$. First use identities and algebra to transform A into some expression C, then use (possibly different) identities and algebra to transform B into the *same* expression C. Conclude that $A = B$.**
5. **Prove that $AD = BC$, with $B \neq 0$ and $D \neq 0$. You can then conclude that $\frac{A}{B} = \frac{C}{D}$.**

There are often a variety of ways to proceed, and it will take some practice before you can easily decide which strategies are likely to be the most efficient in a particular case. Keep these two purposes of working with trigonometric identities in mind:

- to learn the relationships among the trigonometric functions
- to simplify an expression by using an equivalent form

> **CAUTION**
>
> Proving identities is not the same as solving equations. Properties that apply to equations, such as adding the same value to both sides, are not valid when verifying identities because the beginning statement (to be verified) may not be true.

Wow!

In the following example, the Pythagorean identity is used to replace $1 - \sin^2 x$ with $\cos^2 x$. Consider using one of the Pythagorean identities whenever a squared trigonometric function appears.

replace	*with*	*replace*	*with*
$\sin^2 x$	$1 - \cos^2 x$	$\csc^2 x$	$1 + \cot^2 x$
$\cos^2 x$	$1 - \sin^2 x$	$\sec^2 x$	$\tan^2 x + 1$
$\tan^2 x$	$\sec^2 x - 1$	$\cot^2 x$	$\csc^2 x - 1$

Example 3 Transform One Side into the Other Side

Verify that $\dfrac{1 + \sin x - \sin^2 x}{\cos x} = \cos x + \tan x$.

S1

Solution

The graph of each side of the equation is shown in Figure 9.1-2b of Example 1, where it was noted that the equation *might* be an identity. Begin with the left side of the equation.

$$\begin{aligned}
\frac{1 + \sin x - \sin^2 x}{\cos x} &= \frac{(1 - \sin^2 x) + \sin x}{\cos x} && \textit{regrouping terms}\\
&= \frac{\cos^2 x + \sin x}{\cos x} && \textit{Pythagorean identity}\\
&= \frac{\cos^2 x}{\cos x} + \frac{\sin x}{\cos x} && \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}\\
&= \cos x + \frac{\sin x}{\cos x} && \frac{a^2}{a} = a\\
&= \cos x + \tan x && \textit{quotient identity}
\end{aligned}$$

Strategies for proving identities can also be used to simplify complex expressions.

COMMON ERROR ALERT

Students sometimes forget to write exponents for the Pythagorean identities. For example, they might write $\sin x + \cos x = 1$, which is incorrect. To help them remember the exponents, remind them that the Pythagorean Theorem is about *squares* of sides of a right triangle.

Example Notes

The identity in **Example 3** was tested graphically in Example 1b. Now it will be proved (verified).

Example 3 uses a Pythagorean identity to replace $1 - \sin^2 x$ with $\cos^2 x$. Replacements to use in other expressions involving squared trigonometric functions are given above Example 3.

Remind students that an identity is true for all values of the variable for which all terms in the equation are defined. Ask them to describe values of x that would produce an undefined term in the identity in Example 3.
all values of x for which $\cos x = 0$; that is, $x = \frac{\pi}{2} + k\pi$, where k is any integer

ADDITIONAL EXAMPLES

Example 3

Verify that $\dfrac{1 + \cos x - \cos^2 x}{\sin x} = \sin x + \cot x$.

(It was noted in Additional Example 1b that this *might* be an identity.)

$$\begin{aligned}
&\frac{1 + \cos x - \cos^2 x}{\sin x}\\
&= \frac{(1 - \cos^2 x) + \cos x}{\sin x} && \textit{regrouping terms}\\
&= \frac{\sin^2 x + \cos x}{\sin x} && \textit{Pythagorean identity}\\
&= \frac{\sin^2 x}{\sin x} + \frac{\cos x}{\sin x} && \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}\\
&= \sin x + \frac{\cos x}{\sin x} && \frac{a^2}{a} = a\\
&= \sin x + \cot x && \textit{quotient identity}
\end{aligned}$$

Example Notes

After students complete Example 4, point out that they have verified the identity

$(\csc x + \cot x)(1 - \cos x) = \sin x$.

COMMON ERROR ALERT

The following is *not* a correct proof for **Example 5:**

$$\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

$$\frac{\sin x}{1+\cos x}\cdot\frac{1}{1-\cos x} = \frac{1-\cos x}{\sin x}\cdot\frac{1}{1-\cos x}$$

$$\frac{\sin x}{1-\cos^2 x} = \frac{1}{\sin x}$$

$$\frac{\sin x}{\sin^2 x} = \frac{1}{\sin x}$$

$$\frac{1}{\sin x} = \frac{1}{\sin x}$$

Therefore, the original equation is an identity.

The error made above was *assuming the first equation true,* and then multiplying both sides by $\frac{1}{1-\cos x}$. In fact, the first equation *is the equation that is to be proved.* This type of error is addressed in the **CAUTION** on page 575.

Make sure students understand that $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$ *is an identity,* but the work shown above *does not prove it.*

ADDITIONAL EXAMPLES

Example 4

Simplify $(\sec x + \tan x)(1 - \sin x)$.

$(\sec x + \tan x)(1 - \sin x)$

$= \left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)(1 - \sin x)$	reciprocal and quotient identities
$= \left(\frac{1 + \sin x}{\cos x}\right)(1 - \sin x)$	$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$
$= \frac{(1 + \sin x)(1 - \sin x)}{\cos x}$	$\frac{a}{c}\cdot b = \frac{ab}{c}$
$= \frac{1 - \sin^2 x}{\cos x}$	$(a+b)(a-b) = a^2 - b^2$
$= \frac{\cos^2 x}{\cos x}$	Pythagorean identity
$= \cos x$	$\frac{x^2}{x} = x$

Example 4 Write Everything in Terms of Sine and Cosine

Simplify $(\csc x + \cot x)(1 - \cos x)$.

Solution

$(\csc x + \cot x)(1 - \cos x)$

$= \left(\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right)(1 - \cos x)$	*reciprocal and quotient identities*
$= \frac{(1 + \cos x)}{\sin x}(1 - \cos x)$	$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$
$= \frac{(1 + \cos x)(1 - \cos x)}{\sin x}$	$\frac{a}{c}\cdot b = \frac{ab}{c}$
$= \frac{1 - \cos^2 x}{\sin x}$	$(a+b)(a-b) = a^2 - b^2$
$= \frac{\sin^2 x}{\sin x}$	*Pythagorean identity*
$= \sin x$	$\frac{x^2}{x} = x$

The strategies presented above and those to be considered are "plans of attack." By themselves they are not much help unless you also have some techniques for carrying out these plans. In the previous examples, the techniques of basic algebra and the use of known identities were used to change trigonometric expressions into equivalent expressions. There is another technique that is often useful when dealing with fractions.

Rewrite a fraction in equivalent form by multiplying its numerator and denominator by the same quantity.

$(x-y)(x+y)$

x^2-y^2

NOTE If a denominator is of the form $1 + \cos x$, multiplying by $1 - \cos x$ gives $1 - \cos^2 x = \sin^2 x$. Similarly, if a denominator is of the form $1 + \sin x$, multiplying by $1 - \sin x$ gives $1 - \sin^2 x = \cos^2 x$. Compare this with earlier techniques used to rationalize denominators and simplify numbers with complex denominators.

Example 5 Transform One Side into the Other Side

Prove that $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$. want

Solution

Beginning with the left side, multiply the numerator and denominator by $1 - \cos x$.

$$\frac{\sin x}{1+\cos x} = \frac{\sin x}{1+\cos x}\cdot\frac{1-\cos x}{1-\cos x}$$

$$= \frac{\sin x(1-\cos x)}{(1+\cos x)(1-\cos x)}$$

$$= \frac{\sin x(1-\cos x)}{1-\cos^2 x}$$

$$= \frac{\sin x(1-\cos x)}{\sin^2 x} \qquad \textit{Pythagorean identity}$$

$$= \frac{1-\cos x}{\sin x}$$

Alternate Solution

The numerators of the given equation, $\sin x$ and $1 - \cos x$, look similar to the Pythagorean identity—except the squares are missing. So begin with the left side and introduce some squares by multiplying it by $\dfrac{\sin x}{\sin x} = 1$.

$$\begin{aligned}
\frac{\sin x}{1 + \cos x} &= \frac{\sin x}{\sin x} \cdot \frac{\sin x}{1 + \cos x} \\
&= \frac{\sin^2 x}{\sin x(1 + \cos x)} \\
&= \frac{1 - \cos^2 x}{\sin x(1 + \cos x)} && \textit{Pythagorean identity} \\
&= \frac{(1 - \cos x)(1 + \cos x)}{\sin x(1 + \cos x)} && a^2 - b^2 = (a - b)(a + b) \\
&= \frac{1 - \cos x}{\sin x}
\end{aligned}$$

Example 6 Dealing with Each Side Separately

Prove that $\csc x - \cot x = \dfrac{\sin x}{1 + \cos x}$.

Solution

Begin with the left side.

$$\begin{aligned}
\csc x - \cot x &= \frac{1}{\sin x} - \frac{\cos x}{\sin x} \\
&= \frac{1 - \cos x}{\sin x} && [1]
\end{aligned}$$

Example 5 shows that the right side of the identity to be proved can also be transformed into this same expression.

$$\frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x} \qquad [2]$$

Combining the equalities [1] and [2] proves the identity.

$$\csc x - \cot x = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$$

Proving identities involving fractions can sometimes be quite complicated. It often helps to approach a fractional identity indirectly, as in the following example.

ADDITIONAL EXAMPLES

Example 5

Prove that $\dfrac{\cos x}{1 - \sin x} = \dfrac{1 + \sin x}{\cos x}$.

$$\begin{aligned}
&\frac{\cos x}{1 - \sin x} \\
&= \frac{\cos x}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} \\
&= \frac{\cos x(1 + \sin x)}{(1 - \sin x)(1 + \sin x)} \\
&= \frac{\cos x(1 + \sin x)}{1 - \sin^2 x} \\
&= \frac{\cos x(1 + \sin x)}{\cos^2 x} && \textit{Pythagorean identity} \\
&= \frac{1 + \sin x}{\cos x}
\end{aligned}$$

Alternate Solution

$$\begin{aligned}
&\frac{\cos x}{1 - \sin x} \\
&= \frac{\cos x}{\cos x} \cdot \frac{\cos x}{1 - \sin x} \\
&= \frac{\cos^2 x}{\cos x(1 - \sin x)} \\
&= \frac{1 - \sin^2 x}{\cos x(1 - \sin x)} && \textit{Pythagorean identity} \\
&= \frac{(1 + \sin x)(1 - \sin x)}{\cos x(1 - \sin x)} && a^2 - b^2 = (a + b)(a - b) \\
&= \frac{1 + \sin x}{\cos x}
\end{aligned}$$

Example 6

Prove that $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$.

Begin with the left side.

$$\begin{aligned}
\sec x + \tan x &= \frac{1}{\cos x} + \frac{\sin x}{\cos x} \\
&= \frac{1 + \sin x}{\cos x}
\end{aligned}$$

Using Additional Example 5, transform the right side.

$$\frac{\cos x}{1 - \sin x} = \frac{1 + \sin x}{\cos x}$$

Thus,

$$\sec x + \tan x = \frac{1 + \sin x}{\cos x} = \frac{\cos x}{1 - \sin x}.$$

Therefore, $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$.

Example Notes

For **Example 7**, point out that in part **b**, you can divide both sides of the equation $\sec x(\sec x - \cos x) = \tan^2 x$ by $\tan x(\sec x - \cos x)$ because the equation $\sec x(\sec x - \cos x) = \tan^2 x$ was *proved to be an identity* in part **a**.

Teaching Notes

After completing **Examples 7** and **8**, refer students to the paragraphs at the bottom of page 579. Challenge them to develop an alternate proof for Example 7b or 8.

An alternate proof for each is shown below.

Alternate proof for **Example 7b:**

$$\frac{\sec x}{\tan x}$$

$$= \frac{\sec x(\sec x - \cos x)}{\tan x(\sec x - \cos x)} \qquad \frac{a}{a} = 1$$

$$= \frac{\sec^2 x - \sec x \cos x}{\tan x(\sec x - \cos x)} \qquad \textit{distributive property}$$

$$= \frac{\sec^2 x - \frac{1}{\cos x}\cos x}{\tan x(\sec x - \cos x)} \qquad \textit{reciprocal identity}$$

$$= \frac{\sec^2 x - 1}{\tan x(\sec x - \cos x)} \qquad \frac{1}{a} \cdot a = 1$$

$$= \frac{\tan^2 x}{\tan x(\sec x - \cos x)} \qquad \textit{Pythagorean identity}$$

$$= \frac{\tan x}{\sec x - \cos x} \qquad \frac{a^2}{a} = a$$

Alternate proof for **Example 8:**

$$\frac{\cot x - 1}{\cot x + 1}$$

$$= \frac{\tan x(\cot x - 1)}{\tan x(\cot x + 1)} \qquad \frac{a}{a} = 1$$

$$= \frac{\tan x \cot x - \tan x}{\tan x \cot x + \tan x} \qquad \textit{distributive property}$$

$$= \frac{\tan x \frac{1}{\tan x} - \tan x}{\tan x \frac{1}{\tan x} + \tan x} \qquad \textit{reciprocal identity}$$

$$= \frac{1 - \tan x}{1 + \tan x} \qquad a \cdot \frac{1}{a} = 1$$

Example 7 Proving Identities that Involve Fractions

Prove the first identity below, then use the first identity to prove the second identity. $AD = BC \rightarrow \frac{A}{B} = \frac{C}{D}$

a. $\sec x(\sec x - \cos x) = \tan^2 x$ **b.** $\dfrac{\sec x}{\tan x} = \dfrac{\tan x}{\sec x - \cos x}$

Solution

a. Begin by transforming the left side.

$$\begin{aligned}\sec x(\sec x - \cos x) &= \sec^2 x - \sec x \cos x \\ &= \sec^2 x - \frac{1}{\cos x}\cos x && \textit{reciprocal identity} \\ &= \sec^2 x - 1 \\ &= \tan^2 x && \textit{Pythagorean identity}\end{aligned}$$

Therefore, $\sec x(\sec x - \cos x) = \tan^2 x$.

b. By part **a**,

$$\sec x(\sec x - \cos x) = \tan^2 x$$

Divide both sides of this equation by $\tan x(\sec x - \cos x)$.

$$\frac{\sec x(\sec x - \cos x)}{\tan x(\sec x - \cos x)} = \frac{\tan^2 x}{\tan x(\sec x - \cos x)}$$

$$\frac{\sec x(\sec x - \cos x)}{\tan x(\sec x - \cos x)} = \frac{\tan x \tan x}{\tan x(\sec x - \cos x)}$$

$$\frac{\sec x}{\tan x} = \frac{\tan x}{\sec x - \cos x}$$

Look carefully at how identity **b** was proved in Example 7. First prove identity **a**, which is of the form $AD = BC$ (with $A = \sec x$, $B = \tan x$, $C = \tan x$, and $D = \sec x - \cos x$). Then divide both sides by BD, that is, by $\tan x(\sec x - \cos x)$, to conclude that $\frac{A}{B} = \frac{C}{D}$. This property provides a useful strategy for dealing with identities involving fractions.

Example 8 If $AD = BC$, with $B \neq 0$ and $D \neq 0$, then $\frac{A}{B} = \frac{C}{D}$.

Prove that $\dfrac{\cot x - 1}{\cot x + 1} = \dfrac{1 - \tan x}{1 + \tan x}$.

Solution

Use the same strategy used in Example 7. First prove $AD = BC$, with $A = \cot x - 1$, $B = \cot x + 1$, $C = 1 - \tan x$, and $D = 1 + \tan x$.

$$AD = BC$$

$$(\cot x - 1)(1 + \tan x) = (\cot x + 1)(1 - \tan x)$$

[3]

Multiply out the left side of [3].

$$\begin{aligned}(\cot x - 1)(1 + \tan x) &= \cot x + \cot x \tan x - 1 - \tan x\\ &= \cot x + \frac{1}{\tan x}\tan x - 1 - \tan x\\ &= \cot x + 1 - 1 - \tan x\\ &= \cot x - \tan x.\end{aligned}$$

Similarly, multiply the right side of [3].

$$\begin{aligned}(\cot x + 1)(1 - \tan x) &= \cot x - \cot x \tan x + 1 - \tan x\\ &= \cot x - 1 + 1 - \tan x\\ &= \cot x - \tan x.\end{aligned}$$

Because the left and right sides are equal to the same expression, [3] has been proven to be an identity. Therefore, conclude that

$$\frac{\cot x - 1}{\cot x + 1} = \frac{1 - \tan x}{1 + \tan x}$$

is also an identity.

CAUTION

Strategy 5 does *not* say that you begin with a fractional equation $\frac{A}{B} = \frac{C}{D}$ and cross multiply to eliminate the fractions. If you did that, you would be assuming that the statement was true, which is what has to be proved. What the strategy says is that to prove an identity involving fractions, you need only prove a different identity that does not involve fractions. In other words, if you prove that $AD = BC$ whenever $B \neq 0$ and $D \neq 0$, then you can conclude that $\frac{A}{B} = \frac{C}{D}$. Note that you do not *assume* that $AD = BC$; you use some other strategy to *prove* this statement.

It takes a good deal of practice, as well as *much* trial and error, to become proficient at proving identities. The more practice you have, the easier it will become. Because there are many correct methods, your proofs may be quite different from those of your classmates, instructor, or text answers.

If you do not see what to do immediately, try something and see where it leads: multiply out, factor, or multiply numerator and denominator by the same nonzero quantity. Even if this does not lead anywhere, it may give you some ideas on other strategies to try. When you do obtain a proof, check to see if it can be done more efficiently. Do not include the "side trips" in your final proof—they may have given you some ideas, but they are not part of the proof.

ADDITIONAL EXAMPLES

Example 7

Prove the first identity below, then use the first identity to prove the second identity.

a. $\csc x(\csc x - \sin x) = \cot^2 x$

$\csc x(\csc x - \sin x)$

$= \csc^2 x - \csc x \sin x$

$= \csc^2 x - \frac{1}{\sin x}\sin x$ *reciprocal identity*

$= \csc^2 x - 1$

$= \cot^2 x$ *Pythagorean identity*

b. $\frac{\csc x}{\cot x} = \frac{\cot x}{\csc x - \sin x}$

By part **a,** $\csc x(\csc x - \sin x) = \cot^2 x$.

Divide both sides by $\cot x(\csc x - \sin x)$.

$$\frac{\csc x(\csc x - \sin x)}{\cot x(\csc x - \sin x)} = \frac{\cot^2 x}{\cot x(\csc x - \sin x)}$$

$$\frac{\csc x(\csc x - \sin x)}{\cot x(\csc x - \sin x)} = \frac{\cot x \cot x}{\cot x(\csc x - \sin x)}$$

$$\frac{\csc x}{\cot x} = \frac{\cot x}{\csc x - \sin x}$$

Example 8

Prove that $\frac{\cos x - 1}{\cos x + 1} = \frac{1 - \sec x}{1 + \sec x}$.

First prove $(\cos x - 1)(1 + \sec x) = (\cos x + 1)(1 - \sec x)$.

Multiply out the left side.

$$\begin{aligned}&(\cos x - 1)(1 + \sec x)\\ &= \cos x + \cos x \sec x - 1 - \sec x\\ &= \cos x + \cos x \frac{1}{\cos x} - 1 - \sec x\\ &= \cos x + 1 - 1 - \sec x\\ &= \cos x - \sec x\end{aligned}$$

Multiply out the right side.

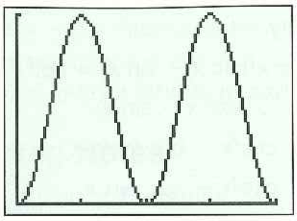
$$\begin{aligned}&(\cos x + 1)(1 - \sec x)\\ &= \cos x - \cos x \sec x + 1 - \sec x\\ &= \cos x - \cos x \frac{1}{\cos x} + 1 - \sec x\\ &= \cos x - 1 + 1 - \sec x\\ &= \cos x - \sec x\end{aligned}$$

Because $(\cos x - 1)(1 + \sec x) = \cos x - \sec x$ and $(\cos x + 1)(1 - \sec x) = \cos x - \sec x$, $(\cos x - 1)(1 + \sec x) = (\cos x + 1)(1 - \sec x)$ is an identity. Therefore, $\frac{\cos x - 1}{\cos x + 1} = \frac{1 - \sec x}{1 + \sec x}$ is also an identity.

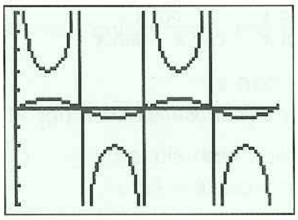
Exercises 9.1

ANSWERS

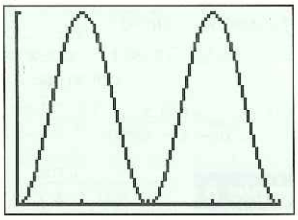
1. possibly an identity



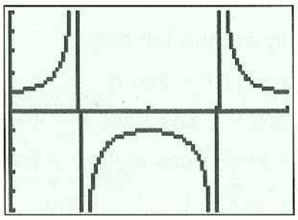
2. not an identity



3. possibly an identity



4. possibly an identity



5. b
6. a
7. e
8. d
9. $\tan x \cos x = \left(\frac{\sin x}{\cos x}\right)\cos x$
   $= \sin x$
10. $\cot x \sin x = \left(\frac{\cos x}{\sin x}\right)\sin x = \cos x$
11. $\cos x \sec x = \cos x\left(\frac{1}{\cos x}\right) = 1$

Exercises 9.1

In Exercises 1–4, test the equation graphically to determine whether it might be an identity. You need not prove those equations that appear to be identities.

1. $\dfrac{\sec x - \cos x}{\sec x} = \sin^2 x$
2. $\tan x + \cot x = \sin x \cos x$
3. $\dfrac{1 - \cos 2x}{2} = \sin^2 x$
4. $\dfrac{\tan x + \cot x}{\csc x} = \sec x$

In Exercises 5–8, insert one of a–f on the right of the equal sign so that the resulting equation appears to be an identity when you test it graphically. You need not prove the identity.

a. $\cos x$ **b.** $\sec x$ **c.** $\sin^2 x$

d. $\sec^2 x$ **e.** $\sin x - \cos x$ **f.** $\dfrac{1}{\sin x \cos x}$

5. $\csc x \tan x =$ ____
6. $\dfrac{\sin x}{\tan x} =$ ____
7. $\dfrac{\sin^4 x - \cos^4 x}{\sin x + \cos x} =$ ____
8. $\tan^2(-x) - \dfrac{\sin(-x)}{\sin x} =$ ____

In Exercises 9–18, prove the identity.

9. $\tan x \cos x = \sin x$
10. $\cot x \sin x = \cos x$
11. $\cos x \sec x = 1$
12. $\sin x \csc x = 1$
13. $\tan x \csc x = \sec x$
14. $\sec x \cot x = \csc x$
15. $\dfrac{\tan x}{\sec x} = \sin x$
16. $\dfrac{\cot x}{\csc x} = \cos x$
17. $(1 + \cos x)(1 - \cos x) = \sin^2 x$
18. $(\csc x - 1)(\csc x + 1) = \cot^2 x$

In Exercises 19–48, state whether or not the equation is an identity. If it is an identity, prove it.

19. $\sin x = \sqrt{1 - \cos^2 x}$
20. $\cot x = \dfrac{\csc x}{\sec x}$
21. $\dfrac{\sin(-x)}{\cos(-x)} = -\tan x$
22. $\tan x = \sqrt{\sec^2 x - 1}$
23. $\cot(-x) = -\cot x$
24. $\sec(-x) = \sec x$
25. $1 + \sec^2 x = \tan^2 x$
26. $\sec^4 x - \tan^4 x = 1 + 2\tan^2 x$
27. $\sec^2 x - \csc^2 x = \tan^2 x - \cot^2 x$
28. $\sec^2 x + \csc^2 x = \sec^2 x \csc^2 x$
29. $\sin^2 x(\cot x + 1)^2 = \cos^2 x(\tan x + 1)^2$
30. $\cos^2 x(\sec x + 1)^2 = (1 + \cos x)^2$
31. $\sin^2 x - \tan^2 x = -\sin^2 x \tan^2 x$
32. $\cot^2 x - 1 = \csc^2 x$
33. $(\cos^2 x - 1)(\tan^2 x + 1) = -\tan^2 x$
34. $(1 - \cos^2 x)\csc x = \sin x$
35. $\tan x = \dfrac{\sec x}{\csc x}$
36. $\dfrac{\cos(-x)}{\sin(-x)} = -\cot x$
37. $\cos^4 x - \sin^4 x = \cos^2 x - \sin^2 x$
38. $\cot^2 x - \cos^2 x = \cos^2 x \cot^2 x$
39. $(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x$
40. $(1 + \tan x)^2 = \sec^2 x$
41. $\dfrac{\sec x}{\csc x} + \dfrac{\sin x}{\cos x} = 2\tan x$
42. $\dfrac{1 + \cos x}{\sin x} + \dfrac{\sin x}{1 + \cos x} = 2\csc x$
43. $\dfrac{\sec x + \csc x}{1 + \tan x} = \csc x$
44. $\dfrac{\cot x - 1}{1 - \tan x} = \dfrac{\csc x}{\sec x}$

12. $\sin x \csc x = \sin x\left(\frac{1}{\sin x}\right) = 1$
13. $\tan x \csc x = \left(\frac{\sin x}{\cos x}\right)\left(\frac{1}{\sin x}\right) = \frac{1}{\cos x} = \sec x$
14. $\sec x \cot x = \left(\frac{1}{\cos x}\right)\left(\frac{\cos x}{\sin x}\right) = \frac{1}{\sin x} = \csc x$
15. $\dfrac{\tan x}{\sec x} = \dfrac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \sin x$
16. $\dfrac{\cot x}{\csc x} = \dfrac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}} = \cos x$
17. $(1 + \cos x)(1 - \cos x) = 1 - \cos^2 x = \sin^2 x$
18. $(\csc x - 1)(\csc x + 1) = \csc^2 x - 1 = \cot^2 x$
19. not an identity
20. $\dfrac{\csc x}{\sec x} = \dfrac{\frac{1}{\sin x}}{\frac{1}{\cos x}} = \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{1}\right)$
    $= \dfrac{\cos x}{\sin x} = \cot x$

45. $\dfrac{1}{\csc x - \sin x} = \sec x \tan x$

46. $\dfrac{1 + \csc x}{\csc x} = \dfrac{\cos^2 x}{1 - \sin x}$

47. $\dfrac{\sin x - \cos x}{\tan x} = \dfrac{\tan x}{\sin x + \cos x}$

48. $\dfrac{\cot x}{\csc x - 1} = \dfrac{\csc x + 1}{\cot x}$

In Exercises 49–52, half of an identity is given. Graph this half in a viewing window with $-2\pi \le x \le 2\pi$ and write a conjecture as to what the right side of the identity is. Then prove your conjecture.

49. $1 - \dfrac{\sin^2 x}{1 + \cos x} = ?$ *Hint:* What familiar function has a graph that looks like this?

50. $\dfrac{1 + \cos x - \cos^2 x}{\sin x} - \cot x = ?$

51. $(\sin x + \cos x)(\sec x + \csc x) - \cot x - 2 = ?$

52. $\cos^3 x(1 - \tan^4 x + \sec^4 x) = ?$

In Exercises 53–66, prove the identity.

53. $\dfrac{1 - \sin x}{\sec x} = \dfrac{\cos^3 x}{1 + \sin x}$

54. $\dfrac{\sin x}{1 - \cot x} = \dfrac{\cos x}{1 - \tan x} = \cos x + \sin x$

55. $\dfrac{\cos x}{1 - \sin x} = \sec x + \tan x$

56. $\dfrac{1 + \sec x}{\tan x + \sin x} = \csc x$

57. $\dfrac{\cos x \cot x}{\cot x - \cos x} = \dfrac{\cot x + \cos x}{\cos x \cot x}$

58. $\dfrac{\cos^3 x - \sin^3 x}{\cos x - \sin x} = 1 + \sin x \cos x$

59. $\log_{10}(\cot x) = -\log_{10}(\tan x)$

60. $\log_{10}(\sec x) = -\log_{10}(\cos x)$

61. $\log_{10}(\csc x + \cot x) = -\log_{10}(\csc x - \cot x)$

62. $\log_{10}(\sec x + \tan x) = -\log_{10}(\sec x - \tan x)$

63. $\tan x - \tan y = -\tan x \tan y(\cot x - \cot y)$

64. $\dfrac{\tan x - \tan y}{\cot x - \cot y} = -\tan x \tan y$

65. $\dfrac{\cos x - \sin y}{\cos y - \sin x} = \dfrac{\cos y + \sin x}{\cos x + \sin y}$

66. $\dfrac{\tan x + \tan y}{\cot x + \cot y} = \dfrac{\tan x \tan y - 1}{1 - \cot x \cot y}$

9.2 Addition and Subtraction Identities

Objectives

- Use the addition and subtraction identities for sine, cosine, and tangent functions
- Use the cofunction identities

Many times, the input, or *argument*, of the sine or cosine function is the sum or difference of two angles, and you may need to simplify the expression. Be careful not to make this common student error.

$$\sin\left(x + \frac{\pi}{6}\right) \text{ is not } \sin x + \sin\frac{\pi}{6}$$

Graphing Exploration

Verify graphically that the expressions above do NOT form an identity by graphing $Y_1 = \sin\left(x + \frac{\pi}{6}\right)$ and $Y_2 = \sin x + \sin\frac{\pi}{6}$.

21. $\dfrac{\sin(-x)}{\cos(-x)} = \dfrac{-\sin x}{\cos x} = -\tan x$

22. not an identity

23. $\cot(-x) = \dfrac{\cos(-x)}{\sin(-x)}$
$= \dfrac{\cos x}{-\sin x} = -\cot x$

24. $\sec(-x) = \dfrac{1}{\cos(-x)}$
$= \dfrac{1}{\cos x} = \sec x$

25. not an identity

26. $\sec^4 x - \tan^4 x$
$= (\sec^2 x - \tan^2 x)(\sec^2 x + \tan^2 x)$
$= 1(\sec^2 x + \tan^2 x)$
$= ((1 + \tan^2 x) + \tan^2 x)$
$= 1 + 2\tan^2 x$

27. $\sec^2 x - \csc^2 x$
$= (1 + \tan^2 x) - (1 + \cot^2 x)$
$= \tan^2 x - \cot^2 x$

28. $\sec^2 x + \csc^2 x = \dfrac{1}{\cos^2 x} + \dfrac{1}{\sin^2 x}$
$= \dfrac{\sin^2 x + \cos^2 x}{\cos^2 x \sin^2 x} = \dfrac{1}{\sin^2 x \cos^2 x}$
$= \left(\dfrac{1}{\sin^2 x}\right)\left(\dfrac{1}{\cos^2 x}\right) = \csc^2 x \sec^2 x$

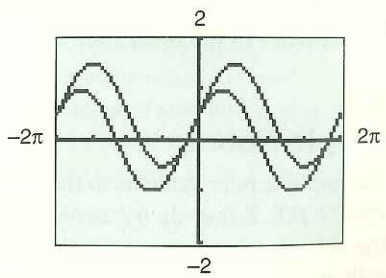
29–66. ***See p. 1070–1072.***

Section 9.2 Addition and Subtraction Identities

Teaching Notes

Students may think $\sin(x + y) = \sin x + \sin y$. This is not true, as shown in the Graphing Exploration.

Solution to the **Graphing Exploration:**



The graphs are different, so $\sin\left(x + \frac{\pi}{6}\right) = \sin x + \sin\frac{\pi}{6}$ is NOT an identity.