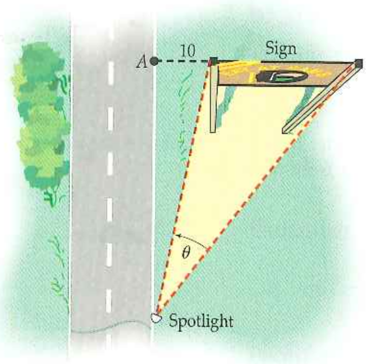
55. a. Let $\cos u = w$ with $0 \le u \le \pi$. Then $u = \cos^{-1} w$, and $\cos^{-1}(\cos u) = \cos^{-1} w = u$. Let $u = \cos^{-1} v$. Then $\cos u = v$, and $\cos(\cos^{-1} v) = \cos u = v$.

b. Let $\tan u = w$ with $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$. Then $u = \tan^{-1} w$, and $\tan^{-1}(\tan u) = \tan^{-1} w = u$. Let $u = \tan^{-1} v$. Then $\tan u = v$, and $\tan(\tan^{-1} v) = \tan u = v$.

56. *Critical Thinking* A 15-foot-wide highway sign is placed 10 feet from a road, perpendicular to the road. A spotlight at the edge of the road is aimed at the sign, as shown in the figure below.



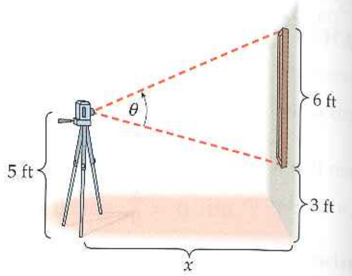
a. Express θ as a function of the distance x from point A to the spotlight.

b. How far from point A should the spotlight be placed so that the angle θ is as large as possible?

a. $\theta = \tan^{-1}\left(\frac{25}{x}\right) - \tan^{-1}\left(\frac{10}{x}\right)$

b. $x \approx 15.8$ feet

57. *Critical Thinking* A camera on a 5-foot-high tripod is placed in front of a 6-foot-high picture that is mounted 3 feet above the floor, as shown in figure below.



a. Express angle θ as a function of the distance x from the camera to the wall.

b. The photographer wants to use a particular lens, for which $\theta = 36°\left(\frac{\pi}{5}\text{ radians}\right)$. How far should she place the camera from the wall to be sure the entire picture will show in the photograph?

a. $\theta = \tan^{-1}\left(\frac{4}{x}\right) + \tan^{-1}\left(\frac{2}{x}\right)$

b. $x \approx 9.13$ feet

Section 8.3

Algebraic Solutions of Trigonometric Equations

Teaching Notes

Point out that because trigonometric functions are periodic functions, there can be an infinite number of solutions to trigonometric equations. In fact, there are an infinite number of solutions to the **basic equations** shown here.

8.3

Algebraic Solutions of Trigonometric Equations

Objective

- Solve trigonometric equations algebraically

Trigonometric equations were solved graphically in Section 8.1. In this section you will learn how to use algebra with inverse trigonometric functions and identities to solve trigonometric equations.

Recall from Section 8.1 that equations such as

$$\sin x = -0.75, \quad \cos x = 0.6, \quad \text{and} \quad \tan x = 3$$

are called **basic equations.** Algebraic solution methods for basic equations are illustrated in Examples 1 through 3.

Example 1 **Solving Basic Cosine Equations**

Solve $\cos x = 0.6$.

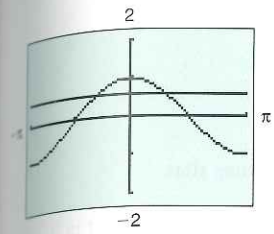


Figure 8.3-1

Solution

The graphs of $Y_1 = \cos x$ and $Y_2 = 0.6$ in Figure 8.3-1 show that there are just two solutions (intersection points) on the interval $[-\pi, \pi]$, which is one full period of the cosine function.

The definition of the inverse cosine function states that

$\cos^{-1}0.6$ is the number in the interval $[0, \pi]$ whose cosine is 0.6.

Using the inverse cosine function, $x = \cos^{-1}0.6 = 0.9273$ is one solution of $\cos x = 0.6$ on the interval $[-\pi, \pi]$. The second solution can be found by using the identity $\cos(-x) = \cos x$, with $x = 0.9273$.

$$\cos(-0.9273) = \cos 0.9273 = 0.6$$

Therefore, the solutions of $\cos x = 0.6$ on the interval $[-\pi, \pi]$ are

$$x = \cos^{-1}0.6 = 0.9273 \quad \text{and} \quad x = -\cos^{-1}0.6 = -0.9273$$

Because the interval $[-\pi, \pi]$ is one complete period of the cosine function, all solutions of $\cos x = 0.6$ are given by

$$x = 0.9273 + 2k\pi \quad \text{and} \quad x = -0.9273 + 2k\pi,$$

where k is any integer.

Example 2 Solving Basic Sine Equations

Solve $\sin x = -0.75$.

Solution

The definition of the inverse sine function states that

$\sin^{-1}(-0.75)$ is the number in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is -0.75.

Using the inverse sine function, $x = \sin^{-1}(-0.75) = -0.8481$ is the solution of $\sin x = -0.75$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. A second solution can be found by using the identity $\sin(\pi - x) = \sin x$, with $x = -0.8481$.

$$\sin[\pi - (-0.8481)] = \sin(3.9897) = -0.75$$

Therefore, $x = \pi - (-0.8481) = 3.9897$ is also a solution of $\sin x = -0.75$, and all solutions are given by

$$x = -0.8481 + 2k\pi \quad \text{and} \quad x = 3.9897 + 2k\pi,$$

where k is any integer.

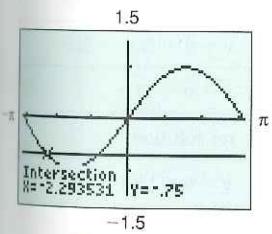


Figure 8.3-2

Recall that there are an infinite number of solutions to many trigonometric equations. Figure 8.3-2 indicates that there are two solutions in the interval $[-\pi, \pi]$: $x = -2.2935$ and $x = -0.8481$. The solution $x = -2.2935$ can be found by letting $k = -1$ in the solution $x = 3.9897 + 2k\pi$.

Example Notes

In **Example 1,** a solution in the interval $[0, \pi]$ is found using the definition of the inverse cosine function. Another solution is found using the identity $\cos(-x) = \cos x$. Both solutions are shown in Figure 8.3-1.

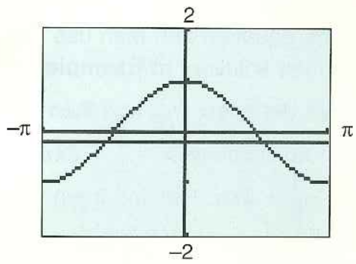
In **Example 2,** a solution in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is found using the definition of the inverse sine function. Another solution is found using the identity $\sin(\pi - x) = \sin x$. Emphasize that the purpose of Figure 8.3-2 is *not* to show these two solutions. Figure 8.3-2 corresponds to the discussion at the bottom of the page. The solution shown in Figure 8.3-2, $x = -2.2935$, is found by letting $k = -1$ in the expression $3.9897 + 2k\pi$.

ADDITIONAL EXAMPLES

Example 1

Solve $\cos x = -0.2$.

The graphs of $Y_1 = \cos x$ and $Y_2 = -0.2$ show that there are 2 solutions in the interval $[-\pi, \pi]$.



The two solutions indicated in the graph are $\cos^{-1}(-0.2) = 1.7722$ and $-\cos^{-1}(-0.2) = -1.7722$.

All solutions are given by $x = 1.7722 + 2k\pi$ and $x = -1.7722 + 2k\pi$, where k is any integer.

Example 2

Solve $\sin x = -0.33$.

Two solutions are $\sin^{-1}(-0.33) = -0.3363$ and $\pi - \sin^{-1}(-0.33) = \pi - (-0.3363) = 3.4779$.

All solutions are given by $x = -0.3363 + 2k\pi$ and $x = 3.4779 + 2k\pi$, where k is any integer.

Teaching Notes

Refer students to the table **Solutions of Basic Trigonometric Equations.** Ask them for examples of the basic equations $\sin x = c$ and $\cos x = c$ that have no solution. Possible answers: $\sin x = 3$ and $\cos x = -1.2$

Remind students that:

- $\sin^{-1} c$ is a number in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ if it is defined.
- $\cos^{-1} c$ is a number in the interval $[0, \pi]$ if it is defined.
- $\tan^{-1} c$ is a number in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Example Notes

For **Example 4,** ask students to show that $x = 1.4455$ is in Quadrant I and $x = -1.4455$ is in Quadrant IV.

$0 < 1.4455 < 1.5708 \approx \frac{\pi}{2}$

$-\frac{\pi}{2} \approx -1.5708 < -1.4455 < 0$

COMMON ERROR ALERT

When solving trigonometric equations, students sometimes isolate a trigonometric function value on one side of the equation and then use that value in their solution. In **Example 4,** they might get $\cos x = \frac{1}{8}$ and then give their solutions as $x = \frac{1}{8} + 2k\pi$ and $x = -\frac{1}{8} + 2k\pi$. Remind them that if $\cos x = \frac{1}{8}$, then x is a number whose cosine is $\frac{1}{8}$; one such number is $x = \cos^{-1}\frac{1}{8} \approx 1.4455$.

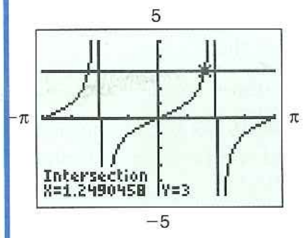


Figure 8.3-3

Example 3 Solving Basic Tangent Equations

Solve $\tan x = 3$.

Solution

The definition of the inverse tangent function states that

$\tan^{-1} 3$ is the number in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is 3.

Using the inverse tangent function, $x = \tan^{-1} 3 = 1.2490$ is the solution of $\tan x = 3$ on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Because $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is one full period of the tangent function, all solutions are given by

$$x = 1.2490 + k\pi,$$

where k is any integer.

The solution method used in Examples 1–3 is summarized in the following table, where k is any integer.

Solutions of Basic Trigonometric Equations

Equation	Possible values of c	Solutions
$\sin x = c$	$-1 < c < 1$	$x = \sin^{-1} c + 2k\pi$ and $x = (\pi - \sin^{-1} c) + 2k\pi$
	$c = 1$	$x = \frac{\pi}{2} + 2k\pi$
	$c = -1$	$x = -\frac{\pi}{2} + 2k\pi$
	$c > 1$ or $c < -1$	no solution
$\cos x = c$	$-1 < c < 1$	$x = \cos^{-1} c + 2k\pi$ and $x = -\cos^{-1} c + 2k\pi$
	$c = 1$	$x = 0 + 2k\pi = 2k\pi$
	$c = -1$	$x = \pi + 2k\pi$
	$c > 1$ or $c < -1$	no solution
$\tan x = c$	all real numbers	$x = \tan^{-1} c + k\pi$

Example 4 Using the Solution Algorithm

Solve $8 \cos x - 1 = 0$.

Solution

First rewrite the equation as an equivalent basic equation.

$$8 \cos x - 1 = 0$$
$$\cos x = \frac{1}{8}$$

Then solve the basic equation using the inverse cosine function. One solution is in Quadrant I.

$$x = \cos^{-1} \frac{1}{8} = 1.4455$$

The other solution on the interval $[-\pi, \pi]$ is in Quadrant IV.

$$x = -\cos^{-1} \frac{1}{8} = -1.4455$$

All solutions are given by

$$x = 1.4455 + 2k\pi \quad \text{and} \quad x = -1.4455 + 2k\pi,$$

where k is any integer.

Example 5 Solving Basic Equations with Special Values

Solve $\sin u = \frac{\sqrt{2}}{2}$ exactly, without using a calculator.

Solution

Because $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $u = \frac{\pi}{4}$ is one solution of $\sin u = \frac{\sqrt{2}}{2}$ on the interval $[-\pi, \pi]$. Another solution is in Quadrant II.

$$u = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Therefore, the exact solution is given by

$$u = \frac{\pi}{4} + 2k\pi \quad \text{and} \quad u = \frac{3\pi}{4} + 2k\pi,$$

where k is any integer.

Sometimes trigonometric equations can be solved by using substitution to make them into basic equations.

ADDITIONAL EXAMPLES

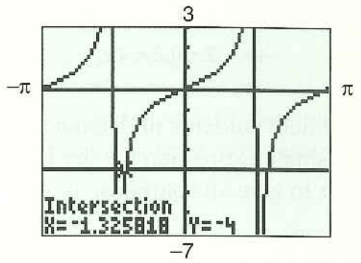
Example 3

Solve $\tan x = -4$.

$\tan^{-1}(-4)$ is the number in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is -4.

$\tan^{-1}(-4) = -1.3258$

All solutions are given by $x = -1.3258 + k\pi$, where k is any integer.



Example 4

Solve $3 \cos x + 1 = 0$.

$3 \cos x + 1 = 0$
$3 \cos x = -1$
$\cos x = -\frac{1}{3}$

The solutions in the interval $[-\pi, \pi]$ are $x = \cos^{-1}\left(-\frac{1}{3}\right) = 1.9106$ and $x = -\cos^{-1}\left(-\frac{1}{3}\right) = -1.9106$.

All solutions are given by $x = 1.9106 + 2k\pi$ and $x = -1.9106 + 2k\pi$, where k is any integer.

Example 5

Solve $\sin u = \frac{1}{2}$ exactly, without using a calculator.

$\sin \frac{\pi}{6} = \frac{1}{2}$, so $u = \frac{\pi}{6}$ is a solution. Another solution is in Quadrant II: $u = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$. Therefore, the exact solutions are given by $u = \frac{\pi}{6} + 2k\pi$ and $u = \frac{5\pi}{6} + 2k\pi$, where k is any integer.

Real-World Application

he substitution technique shown in **xample 6** can be used in finding the ngle of a rifle barrel needed to hit a esired target. See Exercise 56 on ages 546–547.

xample Notes

or **Example 8,** list values for each olution expression as shown below:

$+ k\pi$: $\cdots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \cdots$

$+ 2k\pi$: $\cdots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \cdots$

$+ 2k\pi$: $\cdots, -5\pi, -3\pi, -\pi, \pi, 3\pi, 5\pi, \cdots$

his will help students understand hat the single expression $0 + k\pi$ is ufficient to give all solutions.

DDITIONAL EXAMPLES

Example 6

olve $\sin 4x = \frac{\sqrt{3}}{2}$ exactly, without sing a calculator.

et $u = 4x$ and solve the basic quation $\sin u = \frac{\sqrt{3}}{2}$. The complete olutions of $\sin u = \frac{\sqrt{3}}{2}$ are given by $= \frac{\pi}{3} + 2k\pi$ and $u = \frac{2\pi}{3} + 2k\pi$, here k is any integer.

ubstitute $4x$ for u and solve for x.

$$u = \frac{\pi}{3} + 2k\pi \qquad u = \frac{2\pi}{3} + 2k\pi$$
$$x = \frac{\pi}{3} + 2k\pi \qquad 4x = \frac{2\pi}{3} + 2k\pi$$
$$x = \frac{\pi}{12} + \frac{k\pi}{2} \qquad x = \frac{\pi}{6} + \frac{k\pi}{2}$$

ll solutions are given by $x = \frac{\pi}{12} + \frac{k\pi}{2}$ nd $x = \frac{\pi}{6} + \frac{k\pi}{2}$, where k is any nteger.

Example 7

ind the solutions of $\sin^2 x + 5 \sin x + 1 = 0$ n the interval $[-\pi, \pi]$.

et $u = \sin x$. Rewrite the equation as $u^2 + 5u + 1 = 0$ and solve for u.

$4u + 1)(u + 1) = 0$

$= -\frac{1}{4}$ or $u = -1$

Example 6 Using Substitution and Basic Equations

Solve $\sin 2x = \frac{\sqrt{2}}{2}$ exactly, without using a calculator.

Solution

First, let $u = 2x$, and solve the basic equation $\sin u = \frac{\sqrt{2}}{2}$. From Example 5, you know the complete exact solution of $\sin u = \frac{\sqrt{2}}{2}$ is given by

$$u = \frac{\pi}{4} + 2k\pi \qquad \text{and} \qquad u = \frac{3\pi}{4} + 2k\pi,$$

where k is any integer.

Because $u = 2x$, each of these solutions leads to a solution of the original equation. Substitute $2x$ for u, and solve for x.

$$u = \frac{\pi}{4} + 2k\pi \qquad\qquad u = \frac{3\pi}{4} + 2k\pi$$
$$2x = \frac{\pi}{4} + 2k\pi \qquad \text{and} \qquad 2x = \frac{3\pi}{4} + 2k\pi$$
$$x = \frac{\pi}{8} + k\pi \qquad\qquad x = \frac{3\pi}{8} + k\pi$$

Therefore, all solutions of $\sin 2x = \frac{\sqrt{2}}{2}$ are given by

$$x = \frac{\pi}{8} + k\pi \qquad \text{and} \qquad x = \frac{3\pi}{8} + k\pi,$$

where k is any integer.

Algebraic Techniques

Many trigonometric equations can be solved algebraically—by using factoring, the quadratic formula, and basic identities to write an equivalent equation that involves only basic equations, as shown in the following examples.

Example 7 Factoring Trigonometric Equations

Find the solutions of $3 \sin^2 x - \sin x - 2 = 0$ in the interval $[-\pi, \pi]$.

Solution

Let $u = \sin x$.

$$3 \sin^2 x - \sin x - 2 = 0$$
$$3u^2 - u - 2 = 0 \qquad \textit{Substitution}$$

This quadratic equation can be solved by factoring.

$$3u^2 - u - 2 = 0$$
$$(3u + 2)(u - 1) = 0$$
$$u = -\frac{2}{3} \quad \text{or} \quad u = 1$$

Substituting $\sin x$ for u results in two basic equations.

$$\sin x = -\frac{2}{3} \quad \text{or} \quad \sin x = 1$$

If $\sin x = \left(-\frac{2}{3}\right)$, then

$$x = \sin^{-1}\left(-\frac{2}{3}\right) = -0.7297 + 2k\pi \quad \text{or} \quad x = \pi - \sin^{-1}\left(-\frac{2}{3}\right) = 3.8713 + 2k\pi$$

If $\sin x = 1$, then $x = \frac{\pi}{2} + 2k\pi$

Therefore, the solutions of $3\sin^2 x - \sin x - 2 = 0$ are

$$x = -0.7297 + 2k\pi, \quad x = \frac{\pi}{2} + 2k\pi, \quad \text{and} \quad x = 3.8713 + 2k\pi,$$

where k is any integer.

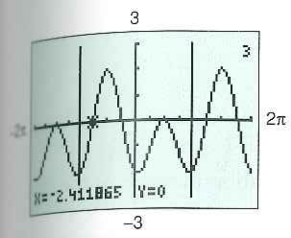


Figure 8.3-4

Figure 8.3-4 indicates that there are three solutions in the interval $[-\pi, \pi]$, which is marked with vertical lines. The solution $x = 3.8713$ is outside the interval, but the corresponding solution within the interval can be found by letting $k = -1$ in $x = 3.8713 + 2k\pi$. Within $[-\pi, \pi]$, the solutions are

$$x = 3.8713 - 2\pi = -2.4119, \quad x = -0.7297, \quad \text{and} \quad x = \frac{\pi}{2}.$$

Example 8 Factoring Trigonometric Equations

Solve $\tan x \cos^2 x = \tan x$.

Solution

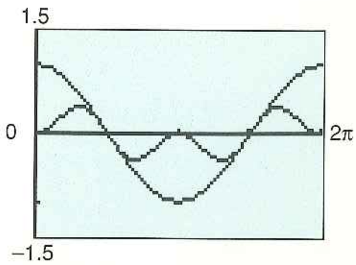
Write an equivalent equation as an expression equal to zero, and factor.

$$\tan x \cos^2 x - \tan x = 0$$
$$\tan x(\cos^2 x - 1) = 0$$

$$\tan x = 0 \quad \text{or} \quad \cos^2 x - 1 = 0$$
$$x = 0 + k\pi \qquad \cos^2 x = 1$$
$$\sqrt{\cos^2 x} = \sqrt{1}$$
$$\cos x = \pm 1$$
$$x = 0 + 2k\pi \quad \text{or} \quad x = \pi + 2k\pi$$

CAUTION

$\cos^2 x - 1 \neq \sin^2 x$



continued from page 542

Substitute $\sin x$ for u and solve for x.

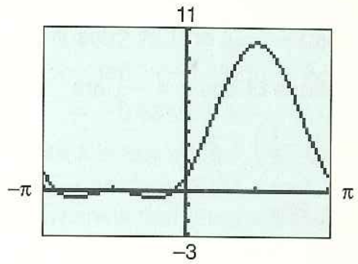
$\sin x = -\frac{1}{4}$ or $\sin x = -1$

Two solutions of $\sin x = -\frac{1}{4}$ are $x = \sin^{-1}\left(-\frac{1}{4}\right) = -0.2527$ and $x = \pi - \sin^{-1}\left(-\frac{1}{4}\right) = 3.3943$.

So all solutions of $\sin x = -\frac{1}{4}$ are given by $x = -0.2527 + 2k\pi$ and $x = 3.3943 + 2k\pi$.

All solutions of $\sin x = -1$ are given by $x = -\frac{\pi}{2} + 2k\pi$.

The figure below indicates that there are three solutions in the interval $[-\pi, \pi]$.



The solutions in the interval $[-\pi, \pi]$ are found by using appropriate values of k as shown below:

Let $k = 0$: $-0.2527 + 2(0)\pi = -0.2527$

Let $k = -1$: $3.3943 + 2(-1)\pi = -2.8889$

Let $k = 0$: $-\frac{\pi}{2} + 2(0)\pi = -\frac{\pi}{2}$

Example 8

Solve $\cos x \sin^2 x = \cos x$.

$\cos x \sin^2 x - \cos x = 0$

$\cos x(\sin^2 x - 1) = 0$

$\cos x = 0$ or $\sin^2 x - 1 = 0$

$\sin^2 x = 1$

$\sin x = \pm 1$

All solutions of:

$\cos x = 0$ are given by $x = \frac{\pi}{2} + k\pi$,

$\sin x = 1$ are given by $x = \frac{\pi}{2} + 2k\pi$,

$\sin x = -1$ are given by $x = -\frac{\pi}{2} + 2k\pi$.

So, all solutions of $\cos x \sin^2 x = \cos x$ are given by $x = \frac{\pi}{2} + k\pi$, where k is any integer.

The graphs of $Y_1 = \cos x \sin^2 x$ and $Y_2 = \cos x$ are shown.

Teaching Notes

The Pythagorean Identities can be found on page 460 or on the inside of the back cover.

ADDITIONAL EXAMPLES

Example 9

Solve $-6\sin^2 x + \cos x + 5 = 0$.

Substitute $1 - \cos^2 x$ for $\sin^2 x$, rewrite the equation in terms of the cosine function, and then solve.

$$-6\sin^2 x + \cos x + 5 = 0$$
$$-6(1 - \cos^2 x) + \cos x + 5 = 0$$
$$-6 + 6\cos^2 x + \cos x + 5 = 0$$
$$6\cos^2 x + \cos x - 1 = 0$$
$$(2\cos x + 1)(3\cos x - 1) = 0$$
$$2\cos x + 1 = 0 \qquad 3\cos x - 1 = 0$$
$$\cos x = -\frac{1}{2} \text{ or } \cos x = \frac{1}{3}$$

The solutions of $\cos x = -\frac{1}{2}$ are

$$x = \cos^{-1}\left(-\frac{1}{2}\right) + 2k\pi$$
$$= \frac{2\pi}{3} + 2k\pi$$

and

$$x = -\cos^{-1}\left(-\frac{1}{2}\right) + 2k\pi$$
$$= -\frac{2\pi}{3} + 2k\pi$$

The solutions of $\cos x = \frac{1}{3}$ are

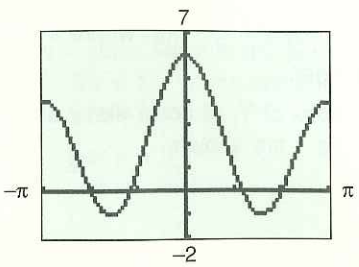
$$x = \cos^{-1}\left(\frac{1}{3}\right) + 2k\pi = 1.2310 + 2k\pi$$

and

$$x = -\cos^{-1}\left(\frac{1}{3}\right) + 2k\pi$$
$$= -1.2310 + 2k\pi.$$

Therefore, all solutions are given by $x = \frac{2\pi}{3} + 2k\pi$, $x = -\frac{2\pi}{3} + 2k\pi$, $x = 1.2310 + 2k\pi$, and $x = -1.2310 + 2k\pi$, where k is any integer.

The graph of $f(x) = -6\sin^2 x + \cos x + 5$ below confirms that there are four solutions in the interval $[-\pi, \pi]$.



More simply stated, the solution of $\tan x \cos^2 x = \tan x$ is

$$x = 0 + k\pi = k\pi,$$

where k is any integer. The graphs of $Y_1 = \tan x \cos^2 x$ and $Y_2 = \tan x$ are shown in Figure 8.3-5.

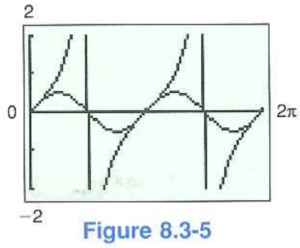


Figure 8.3-5

Many trigonometric equations can be solved if trigonometric identities are used to rewrite the original equation, as shown in Examples 9 and 10.

Example 9 Identities and Factoring

Solve $-10\cos^2 x - 3\sin x + 9 = 0$.

Solution

Use the Pythagorean identity to rewrite the equation in terms of the sine function.

$$-10\cos^2 x - 3\sin x + 9 = 0$$
$$-10(1 - \sin^2 x) - 3\sin x + 9 = 0$$
$$-10 + 10\sin^2 x - 3\sin x + 9 = 0$$
$$10\sin^2 x - 3\sin x - 1 = 0$$

Factor the left side and solve.

$$(2\sin x - 1)(5\sin x + 1) = 0$$
$$2\sin x - 1 = 0 \qquad 5\sin x + 1 = 0$$
$$\sin x = \frac{1}{2} \quad \text{or} \quad \sin x = -\frac{1}{5}$$
$$x = \sin^{-1}\left(\frac{1}{2}\right) \qquad x = \sin^{-1}\left(-\frac{1}{5}\right)$$
$$x = \frac{\pi}{6} + 2k\pi \text{ or} \qquad x = -0.2014 + 2k\pi \text{ or}$$
$$x = \pi - \frac{\pi}{6} + 2k\pi \qquad x = \pi - (-0.2014) + 2k\pi$$
$$= \frac{5\pi}{6} + 2k\pi \qquad = 3.3430 + 2k\pi$$

Therefore, all solutions of $-10\cos^2 x - 3\sin x + 9 = 0$ are

$$x = \frac{\pi}{6} + 2k\pi, \quad x = \frac{5\pi}{6} + 2k\pi,$$
$$x = -0.2014 + 2k\pi, \quad \text{and} \quad x = 3.3430 + 2k\pi,$$

where k is any integer. The graph of $Y_1 = -10\cos^2 x - 3\sin x + 9$ shown in Figure 8.3-6 confirms the solution.

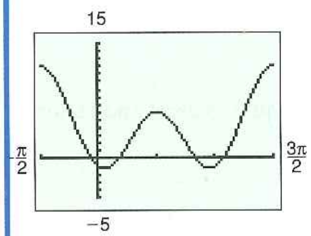


Figure 8.3-6

Example 10 Identities and Quadratic Formula

Solve $\sec^2 x + 5\tan x = -2$.

Solution

Use a Pythagorean identity to rewrite the equation in terms of the tangent function.

$$\sec^2 x + 5\tan x = -2$$
$$\sec^2 x + 5\tan x + 2 = 0$$
$$(1 + \tan^2 x) + 5\tan x + 2 = 0$$
$$\tan^2 x + 5\tan x + 3 = 0$$

Use the quadratic formula to solve for tan *x*.

$$\tan x = \frac{-5 \pm \sqrt{5^2 - 4(1)(3)}}{2(1)} = \frac{-5 \pm \sqrt{13}}{2}$$

$$\tan x = \frac{-5 + \sqrt{13}}{2} = -0.6972 \quad \text{or} \quad \tan x = \frac{-5 - \sqrt{13}}{2} = -4.3028$$

$$x = \tan^{-1}(-0.6972) \qquad x = \tan^{-1}(-4.3028)$$
$$= -0.6088 + k\pi \qquad = -1.3424 + k\pi$$

Therefore, the solution set of $\sec^2 x + 5\tan x = -2$ is

$$x = -0.6089 + \pi k \quad \text{and} \quad x = -1.3424 + \pi k,$$

where *k* is any integer. The graphs of $Y_1 = \sec^2 x + 5\tan x$ and $Y_2 = -2$ in Figure 8.3-7 confirm the solution.

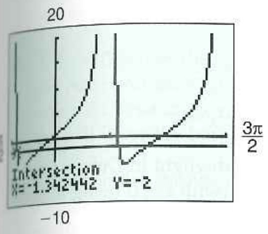


Figure 8.3-7

Exercises 8.3

In Exercises 1–8, find the exact solutions.

1. $\sin x = \frac{\sqrt{3}}{2}$
2. $2\cos x = \sqrt{2}$
3. $\tan x = -\sqrt{3}$
4. $\tan x = 1$
5. $2\cos x = -\sqrt{3}$
6. $\sin x = 0$
7. $2\sin x + 1 = 0$
8. $\csc x = \sqrt{2}$

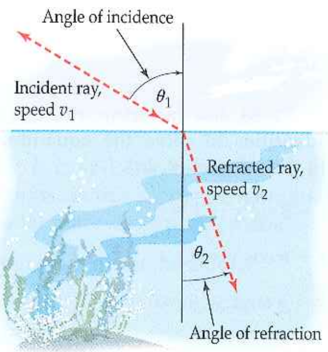
In the following exercises, find exact solutions if possible and approximate solutions otherwise. When a calculator is used, round to four decimal places.

Use the following information in Exercises 9–12.

When a light beam passes from one medium to another (for instance, from air to water), it changes both its speed and direction. According to Snell's Law of Refraction,

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2},$$

where v_1 is the speed of light in the first medium, v_2 its speed in the second medium, θ_1 the angle of incidence, and θ_2 the angle of refraction, as shown in the figure. The number $\frac{v_1}{v_2}$ is called the index of refraction.



9. The index of refraction of light passing from air to water is 1.33. If the angle of incidence is 38°, find the angle of refraction.

ADDITIONAL EXAMPLES

Example 10

Solve $\sec^2 x - 2\tan x = 3$.

Rewrite the equation in terms of the tangent function by substituting $1 + \tan^2 x$ for $\sec^2 x$ and then use the quadratic formula to solve for tan *x*.

$$\sec^2 x - 2\tan x = 3$$
$$\sec^2 x - 2\tan x - 3 = 0$$
$$(1 + \tan^2 x) - 2\tan x - 3 = 0$$
$$\tan^2 x - 2\tan x - 2 = 0$$
$$\tan x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)}$$
$$= \frac{2 \pm \sqrt{12}}{2} = 1 \pm \sqrt{3}$$

$\tan x = 2.7321$ or $\tan x = -0.7321$

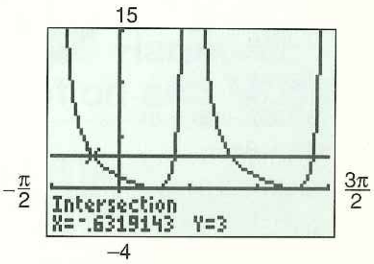
All solutions are given by:

$x = \tan^{-1} 2.7321 + k\pi = 1.2199 + k\pi$

and $x = \tan^{-1}(-0.7321) + k\pi$

$= -0.6319 + k\pi$,

where *k* is any integer. The graphs of $Y_1 = \sec^2 x - 2\tan x$ and $Y_2 = 3$ below show that there are four solutions in the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.



Exercises 8.3

ANSWERS

1. $x = \frac{\pi}{3} + 2k\pi$ or $\frac{2\pi}{3} + 2k\pi$
2. $x = \pm\frac{\pi}{4} + 2k\pi$
3. $x = -\frac{\pi}{3} + k\pi$
4. $x = \frac{\pi}{4} + k\pi$
5. $x = \pm\frac{5\pi}{6} + 2k\pi$
6. $x = k\pi$
7. $x = -\frac{\pi}{6} + 2k\pi$ or $\frac{7\pi}{6} + 2k\pi$
8. $x = \frac{\pi}{4} + 2k\pi$ or $\frac{3\pi}{4} + 2k\pi$
9. 27.5746°

10. The index of refraction of light passing from air to ordinary glass is 1.52. If the angle of incidence is 17°, find the angle of refraction.

11. The index of refraction of light passing from air to dense glass is 1.66. If the angle of incidence is 24°, find the angle of refraction.

12. The index of refraction of light passing from air to quartz is 1.46. If the angle of incidence is 50°, find the angle of refraction.

In Exercises 13–32, find all the solutions of each equation.

13. $\sin x = -0.465$

14. $\sin x = -0.682$

15. $\cos x = -0.564$

16. $\cos x = -0.371$

17. $\tan x = -0.237$

18. $\tan x = -12.45$

19. $\cot x = 2.3 \left[\textit{Hint: } \cot x = \frac{1}{\tan x} \right]$

20. $\cot x = -3.5$

21. $\sec x = -2.65$

22. $\csc x = 5.27$

23. $\sin 2x = -\frac{\sqrt{3}}{2}$

24. $\cos 2x = \frac{\sqrt{2}}{2}$

25. $2 \cos \frac{x}{2} = \sqrt{2}$

26. $2 \sin \frac{x}{3} = 1$

27. $\tan 3x = -\sqrt{3}$

28. $5 \sin 2x = 2$

29. $5 \cos 3x = -3$

30. $2 \tan 4x = 16$

31. $4 \tan \frac{x}{2} = 8$

32. $5 \sin \frac{x}{4} = 4$

In Exercises 33–53, use factoring, the quadratic formula, or identities to solve the equation. Find all solutions in the interval $[0, 2\pi)$.

33. $3 \sin^2 x - 8 \sin x - 3 = 0$

34. $5 \cos^2 x + 6 \cos x = 8$

35. $2 \tan^2 x + 5 \tan x + 3 = 0$

36. $3 \sin^2 x + 2 \sin x = 5$

37. $\cot x \cos x = \cos x$

38. $\tan x \cos x = \cos x$

39. $\cos x \csc x = 2 \cos x$

40. $\tan x \sec x + 3 \tan x = 0$

41. $4 \sin x \tan x - 3 \tan x + 20 \sin x - 15 = 0$
Hint: One factor is $\tan x + 5$.

42. $25 \sin x \cos x - 5 \sin x + 20 \cos x = 4$

43. $\sin^2 x + 2 \sin x - 2 = 0$

44. $\cos^2 x + 5 \cos x = 1$

45. $\tan^2 x + 1 = 3 \tan x$

46. $4 \cos^2 x - 2 \cos x = 1$

47. $2 \tan^2 x - 1 = 3 \tan x$

48. $6 \sin^2 x + 4 \sin x = 1$

49. $\sec^2 x - 2 \tan^2 x = 0$

50. $9 - 12 \sin x = 4 \cos^2 x$

51. $\sec^2 x + \tan x = 3$

52. $\cos^2 x - \sin^2 x + \sin x = 0$

53. $2 \tan^2 x + \tan x = 5 - \sec^2 x$

54. The number of hours of daylight in Detroit on day t of a non-leap year (with $t = 0$ being January 1) is given by the following function.

$$d(t) = 3 \sin\left[\frac{2\pi}{365}(t - 80)\right] + 12$$

   a. On what days of the year are there exactly 11 hours of daylight?
   b. What day has the maximum amount of daylight?

55. A weight hanging from a spring is set into motion moving up and down. Its distance d (in centimeters) above or below the equilibrium point at time t seconds is given by

$$d = 5(\sin 6t - 4 \cos 6t).$$

At what times during the first 2 seconds is the weight at the equilibrium position ($d = 0$)?

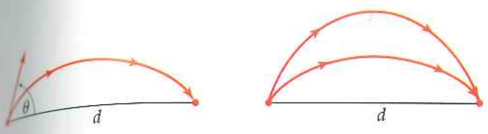
In Exercises 56–59, use the following information.

When a projectile (such as a ball or a bullet) leaves its starting point at angle of elevation θ with velocity v, the horizontal distance d it travels is given by the equation

$$d = \frac{v^2}{32} \sin 2\theta,$$

where d is measured in feet and v in feet per second. Note that the horizontal distance traveled may be the same for two different angles of elevation, so that some of these exercises may have more than one correct answer.

10. 11.0899°
11. 14.1831°
12. 31.6472°
13. $x = -0.4836 + 2k\pi$ or $3.6252 + 2k\pi$
14. $x = -0.7505 + 2k\pi$ or $3.8921 + 2k\pi$
15. $x = \pm 2.1700 + 2k\pi$
16. $x = 1.9509 + 2k\pi$ or $4.3323 + 2k\pi$
17. $x = -0.2327 + k\pi$
18. $x = -1.4906 + k\pi$
19. $x = 0.4101 + k\pi$
20. $x = -0.2783 + k\pi$
21. $x = \pm 1.9577 + 2k\pi$
22. $x = 0.1909 + 2k\pi$ or $2.9507 + 2k\pi$
23. $x = -\frac{\pi}{6} + k\pi$ or $\frac{2\pi}{3} + k\pi$
24. $x = \frac{\pi}{8} + k\pi$ or $\frac{7\pi}{8} + k\pi$
25. $x = \pm\frac{\pi}{2} + 4k\pi$
26. $x = \frac{\pi}{2} + 6k\pi$ or $\frac{5\pi}{2} + 6k\pi$
27. $x = -\frac{\pi}{9} + \frac{k\pi}{3}$
28. $x = 0.2058 + k\pi$ or $1.3650 + k\pi$
29. $x = \pm 0.7381 + \frac{2k\pi}{3}$
30. $x = 0.3616 + \frac{k\pi}{4}$
31. $x = 2.2143 + 2k\pi$
32. $x = 3.7092 + 8k\pi$ or $8.8572 + 8k\pi$
33. $x = 3.4814, 5.9433$
34. $x = 0.6435, 5.6397$
35. $x = \frac{3\pi}{4}, \frac{7\pi}{4}, 2.1588, 5.3004$
36. $x = \frac{\pi}{2}$
37. $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}$
38. $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}$
39. $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$
40. $x = 0, \pi, 1.9106, 4.3726$
41. $x = 0.8481, 1.7682, 2.2935, 4.9098$
42. $x = 1.3694, 4.9137, 4.0689, 5.3559$
43. $x = 0.8213, 2.3203$
44. $x = 1.3770, 4.9062$
45. $x = 0.3649, 1.2059, 3.5065, 4.3475$
46. $x = 0.6283, 1.8850, 4.3982, 5.6549$
47. $x = 1.0591, 2.8679, 4.2007, 6.0095$
48. $x = 0.1949, 2.9466, 5.2472, 4.1776$
49. $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
50. $x = \frac{\pi}{6}, \frac{5\pi}{6}$
51. $x = \frac{\pi}{4}, \frac{5\pi}{4}$
52. $x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2}$
53. $x = \frac{\pi}{4}, \frac{5\pi}{4}$
54. **a.** days 60 and 282
   b. day 171
55. $t = \frac{\tan^{-1} 4}{6} + \frac{k\pi}{6}$
$\approx 1.2682, 0.7446, 0.2210, 1.7918$



a b

56. If muzzle velocity of a rifle is 300 feet per second, at what angle of elevation (in radians) should it be aimed in order for the bullet to hit a target 2500 feet away?

57. Is it possible for the rifle in Exercise 56 to hit a target that is 3000 feet away? At what angle of elevation would it have to be aimed?

58. A fly ball leaves the bat at a velocity of 98 miles per hour and is caught by an outfielder 288 feet away. At what angle of elevation (in degrees) did the ball leave the bat?

59. An outfielder throws the ball at a speed of 75 miles per hour to the catcher who is 200 feet away. At what angle of elevation was the ball thrown?

60. In an alternating current circuit, the voltage is given by the formula

$$V = V_{max} \cdot \sin(2\pi ft + \phi),$$

where V_{max} is the maximum voltage, f is the frequency (in cycles per second), t is the time in seconds, and ϕ is the phase angle.

a. If the phase angle is 0, solve the voltage equation for t.

b. If $\phi = 0$, $V_{max} = 20$, $V = 8.5$, and $f = 120$, find the smallest positive value of t.

61. *Critical Thinking* Find all solutions of $\sin^2 x + 3\cos^2 x = 0$ in the interval $[0, 2\pi)$.

62. *Critical Thinking* What is wrong with this "solution"?

$$\sin x \tan x = \sin x$$
$$\tan x = 1$$
$$x = \frac{\pi}{4} \quad \text{or} \quad \frac{5\pi}{4}$$

Hint: Solve the original equation by moving all terms to one side and factoring. Compare your answers with the ones above.

63. *Critical Thinking* Let n be a fixed positive integer. Describe all solutions of the equation $\sin nx = \frac{1}{2}$.

56. 0.5475 or 1.0233

57. not possible

58. 13.25° or 76.75°

59. 15.97° or 74.03°

60. a. $t = \frac{1}{2\pi f}\sin^{-1}\left(\frac{V}{V_{max}}\right) + \frac{k}{f}$

b. $t = 0.000582$ seconds

61. no solution

62. Since both sides are divided by $\sin x$ there are no solutions that correspond to $\sin x = 0$.

63. $x = \frac{\pi}{6n} + \frac{2k\pi}{n}, \frac{5\pi}{6n} + \frac{2k\pi}{n}$

8.4 Simple Harmonic Motion and Modeling

Objective

- Write a sinusoidal function whose graph resembles a given graph
- Write a sinusoidal function to represent a given simple harmonic motion, and use the function to solve problems
- Find a sinusoidal model for a set of data, and use the model to make predictions

In Section 7.4, graphs of functions of the form

$$f(t) = a\sin(bt + c) + d \quad \text{and} \quad g(t) = a\cos(bt + c) + d,$$

were studied; and the constants a, b, c, and d were examined to see how they affect the graphs of the functions. In this section, trigonometric functions of this form are used to model real-world phenomena.

Recall that if $a \neq 0$ and $b > 0$, then each of the functions

$$f(t) = a\sin(bt + c) + d \quad \text{and} \quad g(t) = a\cos(bt + c) + d$$

has the following characteristics.

$$\text{amplitude} = |a| \qquad \text{period} = \frac{2\pi}{b}$$

$$\text{phase shift} = -\frac{c}{b} \qquad \text{vertical shift} = d$$

Section 8.4 Simple Harmonic Motion and Modeling

Math Background

Functions of the form $f(t) = a\sin(bt + c) + d$ and $g(t) = a\cos(bt + c) + d$ are sinusoidal functions. When two sinusoidal functions are added, the result is a sinusoidal function (see page 511).