In Exercises 15–18, approximate all solutions of the given equation in $(0, 2\pi)$.

15. $\sin x = 0.119$ 16. $\cos x = 0.958$

17. $\tan x = 5$ 18. $\tan x = 17.65$

In Exercises 19–28, find all angles θ with $0° \le \theta < 360°$ that are solutions of the given equation.

19. $\tan \theta = 7.95$ 20. $\tan \theta = 69.4$

21. $\cos \theta = -0.42$ 22. $\cot \theta = -2.4$

23. $2 \sin^2 \theta + 3 \sin \theta + 1 = 0$

24. $4 \cos^2\theta + 4 \cos \theta - 3 = 0$

25. $\tan^2\theta - 3 = 0$ 26. $2 \sin^2\theta = 1$

27. $4 \cos^2\theta + 4 \cos \theta + 1 = 0$
$\theta = 120°, 240°$

28. $\sin^2\theta - 3 \sin \theta = 10$
no solution

At the instant you hear a sonic boom from an airplane overhead, your angle of elevation α to the plane is given by the equation

$$\sin \alpha = \frac{1}{m}$$

where m is the Mach number for the speed of the plane (Mach 1 is the speed of sound, Mach 2.5 is 2.5 times the speed of sound, etc.). In Exercises 29–32, find the angle of elevation (in degrees) for the given Mach number. Remember that an angle of elevation must be between 0° and 90°.

29. $m = 1.1$
$\alpha \approx 65.38°$

30. $m = 1.6$
$\alpha \approx 38.68°$

31. $m = 2$
$\alpha = 30°$

32. $m = 2.4$
$\alpha \approx 24.62°$

33. *Critical Thinking* Under what conditions (on the constant) does a basic equation involving the sine and cosine function have no solutions?

34. *Critical Thinking* Under what conditions (on the constant) does a basic equation involving the secant and cosecant function have no solutions?

15. $x \approx 0.1193$ or 3.0223
16. $x \approx 0.2909$ or $x = 5.9923$
17. $x \approx 1.3734$ or 4.5150
18. $x \approx 1.5142$ or $x = 4.6558$
19. $\theta \approx 82.83°, 262.83°$
20. $\theta \approx 89.17°, 269.17°$
21. $\theta \approx 114.83°, 245.17°$
22. $\theta \approx 337.38°, 157.38°$
23. $\theta = 210°, 270°, 330°$
24. $\theta = 60°, 300°$
25. $\theta = 60°, 120°, 240°, 300°$
26. $\theta = 45°, 135°, 225°, 315°$
33. $\sin x = k$ and $\cos x = k$ have no solutions when $k > 1$ or $k < -1$
34. $\sec x = k$ and $\csc x = k$ have no solutions when $-1 < k < 1$

8.2 Inverse Trigonometric Functions

Objectives

- Define the domain and range of the inverse trigonometric functions
- Use inverse trigonometric function notation

Many trigonometric equations can be solved without graphing. Nongraphical solution methods make use of the *inverse trigonometric functions* that are introduced in this section.

Recall from Section 3.6 that a function cannot have an inverse function unless its graph has the following property.

No horizontal line intersects the graph more than once.

You have seen that the graphs of trigonometric functions do not have this property. However, restricting their domains can modify the trigonometric functions so that they do have inverse functions.

NOTE Other ways of restricting the domains of trigonometric functions are possible. Those presented here for sine, cosine, and tangent are the ones universally agreed upon by mathematicians.

Inverse Sine Function

The *restricted sine function* is $f(x) = \sin x$, when its domain is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Its graph in Figure 8.2-1 shows that for each number v in the interval $[-1, 1]$, there is exactly one number u in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin u = v$.

14. a. The graph of $f(x) = \cos x$ on the interval from 0 to 2π shows that $\cos x = 1$ only when $x = 2\pi$ and 0. Since $\cos x$ has period 2π, all other solutions are obtained by adding or subtracting integer multiples of 2π from 2π, that is, $2\pi + 2\pi = 4\pi$, $2\pi + 2(2\pi) = 6\pi$, $2\pi + 3(2\pi) = 8\pi$, etc., and $2\pi - 2\pi = 0$, $2\pi - 2(2\pi) = -2\pi$, $2\pi - 3(2\pi) = -4\pi$, etc.

b. Similarly, the graph shows that $\cos x = -1$ only when $x = \pi$, so that all solutions are obtained by adding or subtracting integer multiples of 2π from π: $\pi + 2\pi = 3\pi$, $\pi + 2(2\pi) = 5\pi$, $\pi + 3(2\pi) = 7\pi$, etc., and $\pi - 2\pi = -\pi$, $\pi - 2(2\pi) = -3\pi$, $\pi - 3(2\pi) = -5\pi$, etc.

Section 8.2 Inverse Trigonometric Functions

Teaching Notes

Review these facts about inverse functions from page 210:
Let f be a function. The following statements are equivalent.

- The inverse of f is a function.
- f is one-to-one.
- The graph of f passes the Horizontal Line Test.

The inverse function, if it exists, is written as f^{-1}, where if $y = f(x)$, then $x = f^{-1}(y)$.

Point out that trigonometric functions are not one-to-one because different inputs can result in the same output. To illustrate this fact for the sine function, note that $\sin(\pi) = \sin(3\pi) = 0$.

You might want to recall the definition of a one-to-one function (page 208):

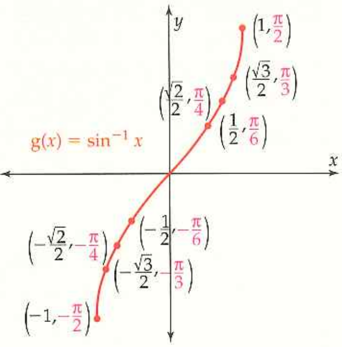
A function f is one-to-one if $f(a) = f(b)$ implies that $a = b$.

Teaching Notes

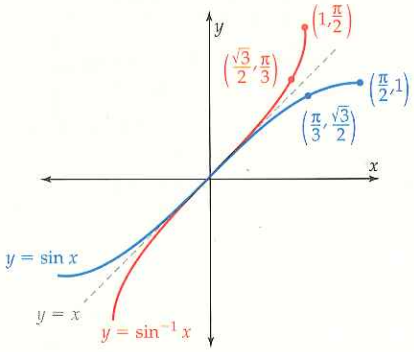
Remind students that the notation $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ represents a closed interval, so it includes its endpoints $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

The range of the **Inverse Sine Function** is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

You might want to have students supply exact function values for the graph of $g(x) = \sin^{-1} x$ shown below. They can use page 448 for reference.



Use a diagram such as the one below to show the symmetric relationship of the restricted sine graph and the inverse sine graph.



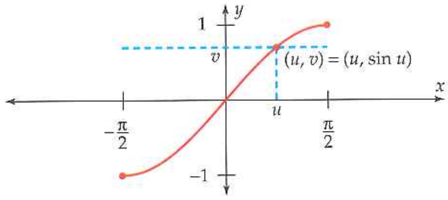


Figure 8.2-1

Because the graph of the restricted sine function passes the Horizontal Line Test, it has an inverse function. This inverse function is called the **inverse sine** (or **arcsine**) **function** and is denoted by

$$g(x) = \sin^{-1} x \text{ or } g(x) = \arcsin x.$$

It is convenient to think of a value of an inverse trigonometric function as an angle; $\sin^{-1} \frac{\sqrt{3}}{2}$ represents an angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $\frac{\sqrt{3}}{2}$. Since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, then $\sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}$.

The graph of the inverse sine function, shown in Figure 8.2-2, is readily obtained from a calculator. Because $g(x) = \sin^{-1} x$ is the inverse of the restricted sine function, its graph is the reflection of the graph of the restricted sine function across the line $y = x$.

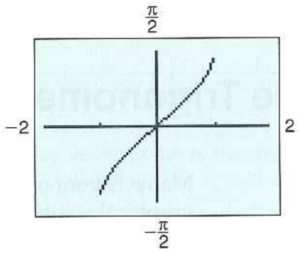


Figure 8.2-2

The domain of $g(x) = \sin^{-1} x$ is the interval $[-1, 1]$, and its range is the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Inverse Sine Function

For each v with $-1 \le v \le 1$,

$\sin^{-1} v$ is the unique number u in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is v; that is,

$$\sin^{-1} v = u \quad \text{exactly when} \quad \sin u = v.$$

Technology Tip

Unless otherwise noted, make sure your calculator is in radian mode.

Technology Tip

If you attempt to use a calculator to evaluate the inverse sine function at a number not in its domain, such as $\sin^{-1}(2)$, you will get an error message.

The inverse sine function can be evaluated by using the SIN^{-1} key (sometimes labeled ASIN) on a calculator. For example,

$$\sin^{-1}(-0.67) = -0.7342 \quad \text{and} \quad \sin^{-1} 0.42 = 0.4334.$$

For many special values, however, you can evaluate the inverse sine function without using a calculator.

Example 1 **Special Values**

Evaluate:

a. $\sin^{-1}\frac{1}{2}$ **b.** $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Solution

a. $\sin^{-1}\frac{1}{2}$ is the number in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $\frac{1}{2}$. From your study of special values, you know that $\sin\frac{\pi}{6} = \frac{1}{2}$. Because $\frac{\pi}{6}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$.

b. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$ because $\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ *and* $-\frac{\pi}{4}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

CAUTION

The notation $\sin^{-1} x$ is *not* exponential notation. It does *not* mean $(\sin x)^{-1}$ or $\frac{1}{\sin x}$. For instance, Example 1 shows that

$$\sin^{-1}\frac{1}{2} = \frac{\pi}{6} \approx 0.5236,$$

but this is not equivalent to

$$\left(\sin\frac{1}{2}\right)^{-1} = \frac{1}{\sin\frac{1}{2}} \approx \frac{1}{0.4794} \approx 2.0858.$$

Suppose $-1 \le v \le 1$ and $\sin^{-1} v = u$. Then by definition of the inverse sine function, $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$ and $\sin u = v$. Therefore,

$$\sin^{-1}(\sin u) = \sin^{-1}(v) = u \qquad \text{and} \qquad \sin(\sin^{-1} v) = \sin(u) = v.$$

This shows that the restricted sine function and the inverse sine function have the usual composition properties of other inverse functions.

Example Notes

Finding the inverse trigonometric function values in **Example 1** depends on knowing special trigonometric function values.

In **1a,** students can recall the appropriate angle in standard position whose sine is $\frac{1}{2}$. In **1b,** students can recall the appropriate angle in standard position whose sine is $-\frac{\sqrt{2}}{2}$, or $-\frac{1}{\sqrt{2}}$.

ADDITIONAL EXAMPLES

Example 1

a. Evaluate $\sin^{-1}\frac{\sqrt{3}}{2}$. $\frac{\pi}{3}$

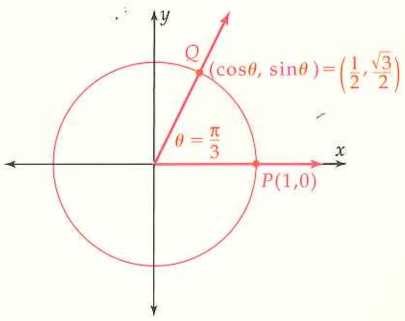
b. Evaluate $\sin^{-1}(-1)$. $-\frac{\pi}{2}$

Teaching Notes

Discuss the second **Technology Tip** on page 531. Challenge students to find all values of x for which they will get an error message if they try to compute $\sin^{-1} x$. $x < -1$, $x > 1$

Math Background

The inverse sine function is also called the arcsine function because an inverse sine function value is an arc measure. For example, $\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$. Note in the unit circle below, $\frac{\pi}{3}$ is the measure of arc PQ.



eaching Notes

o illustrate the Properties of Inverse ine, ask:

What is the sine of $\frac{\pi}{2}$? 1

What is the inverse sine of 1? $\frac{\pi}{2}$

ummarize these two facts by saying:

The inverse sine of the sine of $\frac{\pi}{2}$ is $\frac{\pi}{2}$."

nd

The sine of the inverse sine of 1 is 1."

xample Notes

n Example 2, students can use ecimal values to show that is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but $\frac{5\pi}{6}$ is ot in that interval.

$\approx 0.52, \frac{5\pi}{6} \approx 2.62, \frac{\pi}{2} \approx 1.57$

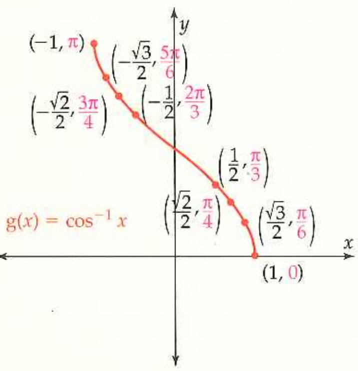
.52 is in $[-1.57, 1.57]$, but

.62 is not in $[-1.57, 1.57]$.

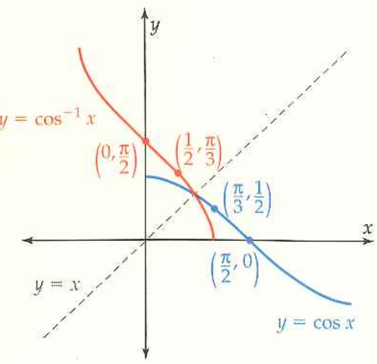
eaching Notes

he range of the inverse cosine unction is $[0, \pi]$.

ou might want to have students upply exact function values for the raph of $g(x) = \cos^{-1} x$ shown below. hey can use page 448 for reference.



Jse a diagram such as the one below o show the symmetric relationship of he restricted cosine graph and the nverse cosine graph.



Properties of Inverse Sine

$$\sin^{-1}(\sin u) = u \quad \text{if} \quad -\frac{\pi}{2} \le u \le \frac{\pi}{2}$$

$$\sin(\sin^{-1} v) = v \quad \text{if} \quad -1 \le v \le 1$$

Example 2 Composition of Inverse Functions

Explain why $\sin^{-1}\left(\sin \frac{\pi}{6}\right) = \frac{\pi}{6}$ is true but $\sin^{-1}\left(\sin \frac{5\pi}{6}\right) = \frac{5\pi}{6}$ is *not* true.

Solution

You know that $\sin \frac{\pi}{6} = \frac{1}{2}$, so by substitution

$$\sin^{-1}\left(\sin \frac{\pi}{6}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

because $\frac{\pi}{6}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Although $\sin \frac{5\pi}{6}$ is also $\frac{1}{2}$, by substitution

$$\sin^{-1}\left(\sin \frac{5\pi}{6}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6},$$

not $\frac{5\pi}{6}$, because $\frac{5\pi}{6}$ is not in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Inverse Cosine Function

The *restricted cosine function* is $f(x) = \cos x$, when its domain is restricted to the interval $[0, \pi]$. Its graph in Figure 8.2-3 shows that for each number v in the interval $[-1, 1]$, there is exactly one number u in the interval $[0, \pi]$ such that $\cos u = v$.

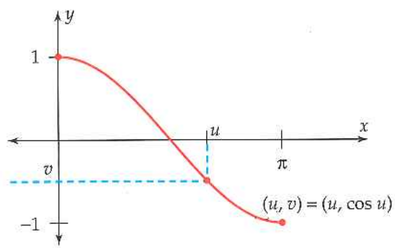


Figure 8.2-3

Because the graph of the restricted cosine function passes the horizontal line test, it has an inverse function. This inverse function is called the **inverse cosine** (or **arccosine**) **function** and is denoted by

$$g(x) = \cos^{-1} x \text{ or } g(x) = \arccos x.$$

The graph of the inverse cosine function, which is the reflection of the graph of the restricted cosine function (Figure 8.2-3) across the line $y = x$, is shown in Figure 8.2-4.

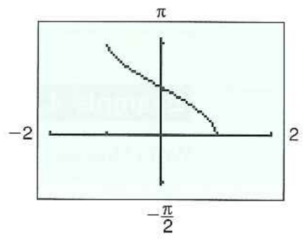


Figure 8.2-4

The domain of $g(x) = \cos^{-1} x$ is the interval $[-1, 1]$ and its range is $[0, \pi]$.

Inverse Cosine Function

> **For each v with $-1 \le v \le 1$,**
>
> **$\cos^{-1} v$ is the unique number u in the interval $[0, \pi]$ whose cosine is v; that is,**
>
> $$\cos^{-1} v = u \quad \text{exactly when} \quad \cos u = v.$$

The properties of the inverse cosine function are similar to the properties of the inverse sine function.

Properties of Inverse Cosine

> $$\cos^{-1}(\cos u) = u \quad \text{if} \quad 0 \le u \le \pi$$
> $$\cos(\cos^{-1} v) = v \quad \text{if} \quad -1 \le v \le 1$$

Example 3 Evaluating Inverse Cosine Expressions

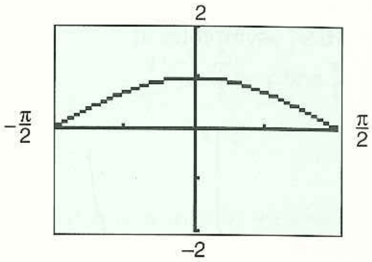
Evaluate the following.

a. $\cos^{-1} \frac{1}{2}$ **b.** $\cos^{-1} 0$ **c.** $\cos^{-1}(-0.63)$

Teaching Notes

Some students may wonder why the domain of the restricted cosine function is $[0, \pi]$ instead of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Have students graph the cosine function with the viewing window $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and explain why that interval is not the domain.



This graph does not pass the Horizontal Line Test.

ADDITIONAL EXAMPLES

Example 2

Explain why $\sin^{-1}\left(\sin \frac{\pi}{4}\right) = \frac{\pi}{4}$ is true, but $\sin^{-1}\left(\sin \frac{7\pi}{4}\right) = \frac{7\pi}{4}$ is *not* true.

$\sin^{-1}\left(\sin \frac{\pi}{4}\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ because $\frac{\pi}{4}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\sin^{-1}\left(\sin \frac{7\pi}{4}\right) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$, not $\frac{7\pi}{4}$, because $\frac{7\pi}{4}$ is not in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Example 3

Evaluate the following.

a. $\cos^{-1} 1$ 0

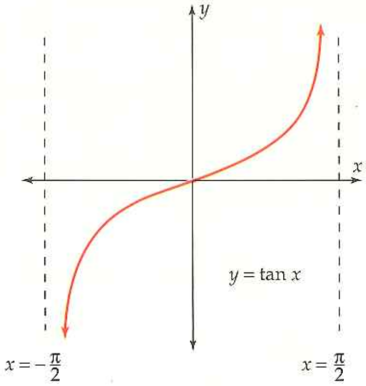
b. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ $\frac{3\pi}{4}$

c. $\cos^{-1}(0.14)$ 1.4303

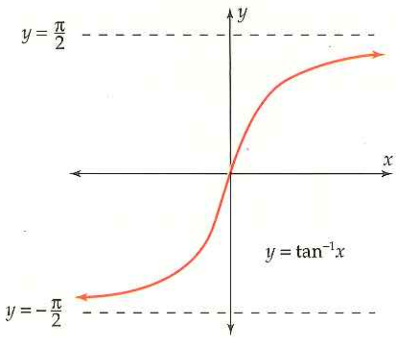
Teaching Notes

The tangent function is discontinuous at all values of x equal to $\frac{k\pi}{2}$, where k is any integer. Therefore, the domain of the restricted tangent function is the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, not the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The restricted tangent function has vertical asymptotes at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.



The inverse tangent function has horizontal asymptotes at $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$.



The range of the restricted tangent function is all real numbers. Therefore, the domain of the inverse tangent function is all real numbers.

The range of the inverse tangent function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

CAUTION

$\cos^{-1}$ does *not* mean $(\cos x)^{-1}$ or $\frac{1}{\cos x}$.

Solution

a. $\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$ because $\frac{\pi}{3}$ is the unique number in the interval $[0, \pi]$ whose cosine is $\frac{1}{2}$.

b. $\cos^{-1} 0 = \frac{\pi}{2}$ because $\cos\frac{\pi}{2} = 0$ and $0 \le \frac{\pi}{2} \le \pi$.

c. The COS^{-1} command on a calculator shows that $\cos^{-1}(-0.63) = 2.2523$.

Example 4 Equivalent Algebraic Expressions

Write $\sin(\cos^{-1} v)$ as an algebraic expression in v.

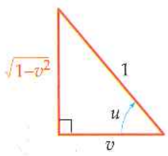


Figure 8.2-5

Solution

Let $\cos^{-1} v = u$, where $0 \le u \le \pi$. Construct a right triangle containing an angle of u radians where $\cos u = \frac{\text{adjacent}}{\text{hypotenuse}} = v$, as shown in Figure 8.2-5. By the Pythagorean Theorem, the length of the side opposite u is $\sqrt{1 - v^2}$. By the definition of sine,

$$\sin u = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{1 - v^2}}{1} = \sqrt{1 - v^2}$$

Therefore, $\sin(\cos^{-1} v) = \sqrt{1 - v^2}$.

Inverse Tangent Function

The *restricted tangent function* is $f(x) = \tan x$, when its domain is restricted to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Its graph in Figure 8.2-6 shows that for every real number v, there is exactly one number u between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that $\tan u = v$.

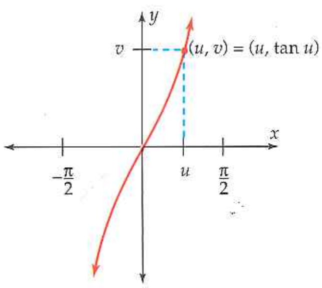


Figure 8.2-6

Because the graph of the restricted tangent function passes the horizontal line test, it has an inverse function. This inverse function is called the **inverse tangent** (or **arctangent**) **function** and is denoted

$$g(x) = \tan^{-1} x \text{ or } g(x) = \arctan x.$$

The graph of the inverse tangent function, which is the reflection of the graph of the restricted tangent function (Figure 8.2-6) across the line $y = x$, is shown in Figure 8.2-7.

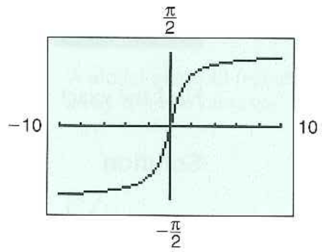


Figure 8.2-7

The domain of $g(x) = \tan^{-1} x$ is the set of all real numbers and its range is the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Inverse Tangent Function

For each real number v,

$\tan^{-1} v$ is the unique number u in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is v; that is,

$$\tan^{-1} v = u \quad \text{exactly when} \quad \tan u = v.$$

The properties of the inverse tangent function are similar to the properties of the inverse sine and inverse cosine functions.

Properties of Inverse Tangent

$$\tan^{-1}(\tan u) = u \quad \text{if} \quad -\frac{\pi}{2} < u < \frac{\pi}{2}$$

$$\tan(\tan^{-1} v) = v \text{ for every real number } v.$$

CAUTION

$\tan^{-1}$ does *not* mean $(\tan x)^{-1}$ or $\frac{1}{\tan x}$.

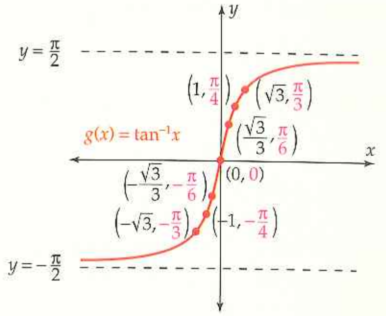
Example 5 Evaluating Inverse Tangent Expressions

Evaluate:

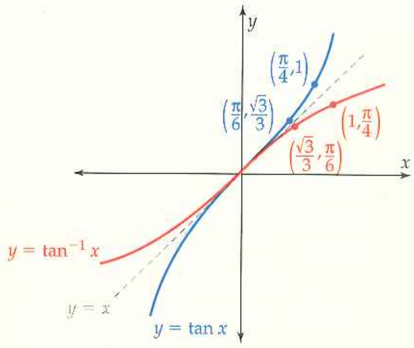
a. $\tan^{-1} 1$ **b.** $\tan^{-1} 136$

Teaching Notes

You might want to have students supply exact function values for the graph of $g(x) = \tan^{-1} x$ shown below. They can use page 448 for reference.



Use a diagram such as the one below to show the symmetric relationship of the restricted tangent graph and the inverse tangent graph.



Example 4, in conjunction with **Additional Example 4,** proves the identity $\sin(\cos^{-1} v) = \cos(\sin^{-1} v)$.

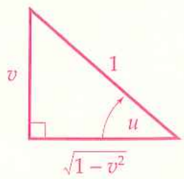
ADDITIONAL EXAMPLES

Example 4

Write $\cos(\sin^{-1} v)$ as an algebraic expression in v.

Let $\sin^{-1} v = u$, where $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$.

Sketch the triangle below showing $\sin u = \frac{\text{opposite}}{\text{hypotenuse}} = v$.



$$\cos u = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{1 - v^2}}{1} = \sqrt{1 - v^2}$$

Therefore, $\cos(\sin^{-1} v) = \sqrt{1 - v^2}$.

Example 5

Evaluate:

a. $\tan^{-1}(-1)$ $-\frac{\pi}{4}$

b. $\tan^{-1}(-14)$ -1.4995

Solution

a. $\tan^{-1} 1 = \frac{\pi}{4}$ because $\frac{\pi}{4}$ is the unique number in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan \frac{\pi}{4} = 1$.

b. The TAN^{-1} key on a calculator shows that $\tan^{-1}(136) = 1.5634$.

Example 6 Exact Values

Find the exact value of $\cos\left(\tan^{-1} \frac{\sqrt{5}}{2}\right)$.

Solution

Let $\tan^{-1} \frac{\sqrt{5}}{2} = u$. Then $\tan u = \frac{\sqrt{5}}{2}$ and $-\frac{\pi}{2} < u < \frac{\pi}{2}$.

Because $\tan u = \frac{\sqrt{5}}{2}$ is positive, u must be between 0 and $\frac{\pi}{2}$. Draw a right triangle containing an angle of u radians whose tangent is $\frac{\sqrt{5}}{2}$.

$$\tan u = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{5}}{2}$$

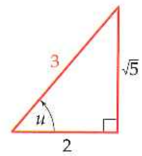


Figure 8.2-8

The hypotenuse has length $\sqrt{2^2 + (\sqrt{5})^2} = \sqrt{4 + 5} = 3$. Therefore,

$$\cos\left(\tan^{-1} \frac{\sqrt{5}}{2}\right) = \cos u = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2}{3}.$$

Exercises 8.2

In Exercises 1–14, find the exact value without using a calculator.

1. $\sin^{-1} 1$ $\frac{\pi}{2}$
2. $\cos^{-1} 0$ $\frac{\pi}{2}$
3. $\tan^{-1}(-1)$ $-\frac{\pi}{4}$
4. $\sin^{-1}(-1)$ $-\frac{\pi}{2}$
5. $\cos^{-1} 1$ 0
6. $\tan^{-1} 1$ $\frac{\pi}{4}$
7. $\tan^{-1} \frac{\sqrt{3}}{3}$ $\frac{\pi}{6}$
8. $\cos^{-1} \frac{\sqrt{3}}{2}$ $\frac{\pi}{6}$
9. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ $-\frac{\pi}{4}$
10. $\sin^{-1} \frac{\sqrt{3}}{2}$ $\frac{\pi}{3}$
11. $\tan^{-1}(-\sqrt{3})$ $-\frac{\pi}{3}$
12. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ $\frac{3\pi}{4}$
13. $\cos^{-1}\left(-\frac{1}{2}\right)$ $\frac{2\pi}{3}$
14. $\sin^{-1}\left(-\frac{1}{2}\right)$ $-\frac{\pi}{6}$

In Exercises 15–24, use a calculator in radian mode to approximate the functional value.

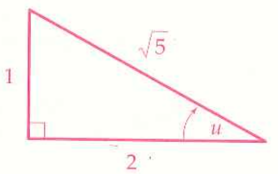
15. $\sin^{-1} 0.35$ **0.3576**
16. $\cos^{-1} 0.76$ **0.7075**
17. $\tan^{-1}(-3.256)$ **−1.2728**
18. $\sin^{-1}(-0.795)$ **−0.9190**

ADDITIONAL EXAMPLES

Example 6

Find the exact value of $\sin\left(\tan^{-1} \frac{1}{2}\right)$.

Let $\tan^{-1} \frac{1}{2} = u$. Then $\tan u = \frac{1}{2}$, where $-\frac{\pi}{2} < u < \frac{\pi}{2}$. Because $\tan u = \frac{1}{2}$ is positive, u is between 0 and $\frac{\pi}{2}$. In a right triangle, $\tan u = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{2}$. So by the Pythagorean theorem, the hypotenuse of the triangle has length $\sqrt{1^2 + 2^2} = \sqrt{5}$.



Therefore, $\sin\left(\tan^{-1} \frac{1}{2}\right) = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$.

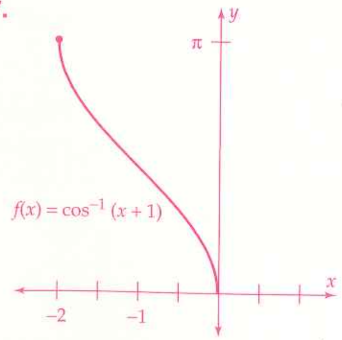
Real-World Application

Inverse trigonometric functions are needed to solve many application problems. An example is Exercise 51 (page 537), in which the goal is to find an angle of elevation. For part b, students will need to evaluate $\sin^{-1}\left(\frac{40}{x}\right)$ when $x = 250$.

Exercises 8.2

ANSWERS

47.



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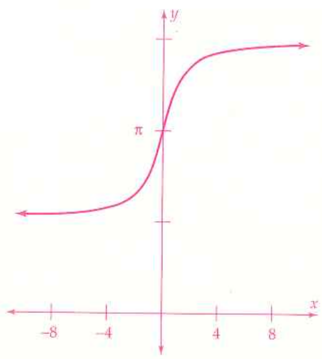

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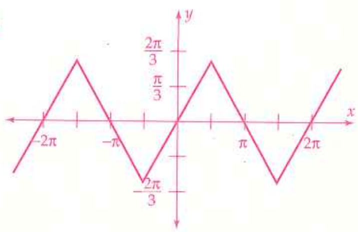


19. $\sin^{-1}(\sin 7)$ *Hint:* the answer is not 7. 0.7168

20. $\cos^{-1}(\cos 3.5)$ 2.7832 21. $\tan^{-1}[\tan(-4)]$ -0.8584

22. $\sin^{-1}[\sin(-2)]$ -1.1416 23. $\cos^{-1}[\cos(-8.5)]$ 2.2168

24. $\tan^{-1}(\tan 12.4)$ -0.1664

25. Given that $u = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, find the exact value of $\cos u$ and $\tan u$. $\cos u = \frac{1}{2}; \tan u = -\sqrt{3}$

26. Given that $u = \tan^{-1}\left(\frac{4}{3}\right)$, find the exact value of $\sin u$ and $\sec u$. $\sin u = \frac{4}{5}, \sec u = \frac{5}{3}$

In Exercises 27–42, find the exact functional value without using a calculator.

27. $\sin^{-1}(\cos 0)$ $\frac{\pi}{2}$ 28. $\cos^{-1}\left(\sin\frac{\pi}{6}\right)$ $\frac{\pi}{3}$

29. $\cos^{-1}\left(\sin\frac{4\pi}{3}\right)$ $\frac{5\pi}{6}$ 30. $\tan^{-1}(\cos \pi)$ $-\frac{\pi}{4}$

31. $\sin^{-1}\left(\cos\frac{7\pi}{6}\right)$ $-\frac{\pi}{3}$ 32. $\cos^{-1}\left(\tan\frac{7\pi}{4}\right)$ π

33. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ (See Exercise 19.) $\frac{\pi}{3}$

34. $\cos^{-1}\left(\cos\frac{5\pi}{4}\right)$ $\frac{3\pi}{4}$

35. $\cos^{-1}\left[\cos\left(-\frac{\pi}{6}\right)\right]$ $\frac{\pi}{6}$

36. $\tan^{-1}\left[\tan\left(-\frac{4\pi}{3}\right)\right]$ $-\frac{\pi}{3}$

37. $\sin\left[\cos^{-1}\left(\frac{3}{5}\right)\right]$ (See Example 6.) $\frac{4}{5}$

38. $\tan\left[\sin^{-1}\left(\frac{3}{5}\right)\right]$ $\frac{3}{4}$ 39. $\cos\left[\tan^{-1}\left(-\frac{3}{4}\right)\right]$ $\frac{4}{5}$

40. $\cos\left[\sin^{-1}\left(\frac{\sqrt{3}}{5}\right)\right]$ $\frac{\sqrt{22}}{5}$ 41. $\tan\left[\sin^{-1}\left(\frac{5}{13}\right)\right]$ $\frac{5}{12}$

42. $\sin\left[\cos^{-1}\left(\frac{\sqrt{3}}{13}\right)\right]$ $\frac{\sqrt{166}}{13}$

In Exercises 43–46, write the expression as an algebraic expression in v, as in Example 4.

43. $\cos(\sin^{-1} v)$ $\sqrt{1-v^2}$ 44. $\cot(\cos^{-1} v)$ $\frac{v}{\sqrt{1-v^2}}$

45. $\tan(\sin^{-1} v)$ $\frac{v}{\sqrt{1-v^2}}$ 46. $\sin(2\sin^{-1} v)$ $2v\sqrt{1-v^2}$

In Exercises 47–50, graph the function.

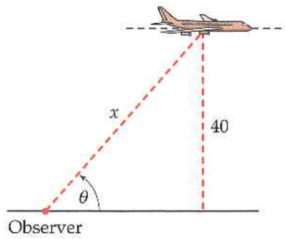
47. $f(x) = \cos^{-1}(x+1)$

48. $g(x) = \tan^{-1} x + \pi$

49. $h(x) = \sin^{-1}(\sin x)$

50. $k(x) = \sin(\sin^{-1} x)$

51. A model plane 40 feet above the ground is flying away from an observer.



a. Express the angle of elevation θ of the plane as a function of the distance x from the observer to the plane. $\theta = \sin^{-1}\left(\frac{40}{x}\right)$

b. What is θ when the plane is 250 feet from the observer? $\approx 9.2°$

52. Show that the restricted secant function, whose domain consists of all numbers x such that $0 \le x \le \pi$ and $x \ne \frac{\pi}{2}$, has an inverse function. Sketch its graph.

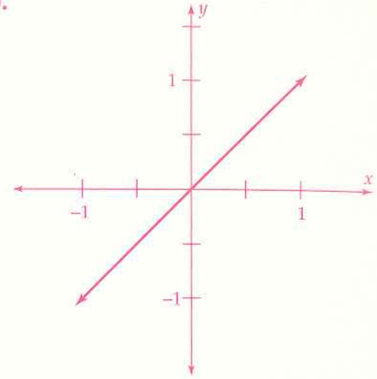
53. Show that the restricted cosecant function, whose domain consists of all numbers x such that $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and $x \ne 0$, has an inverse function. Sketch its graph.

54. Show that the restricted cotangent function, whose domain is the interval $(0, \pi)$, has an inverse function. Sketch its graph.

55. **a.** Show that the inverse cosine function actually has the two properties listed in the box on page 533.

b. Show that the inverse tangent function actually has the two properties listed in the box on page 535.

50.



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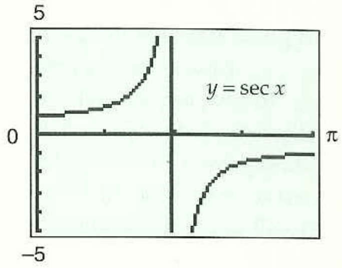
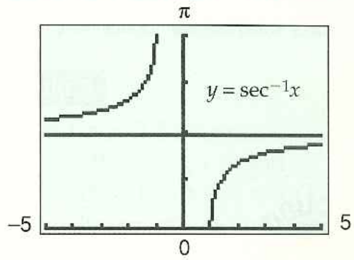


The graph of $y = \sec x$ is one-to-one and has an inverse.



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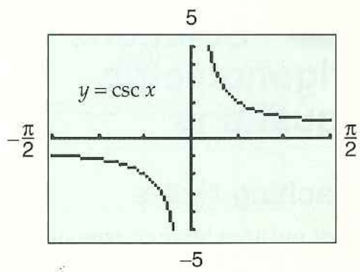
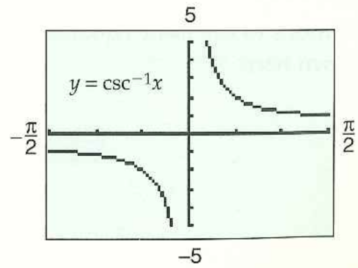
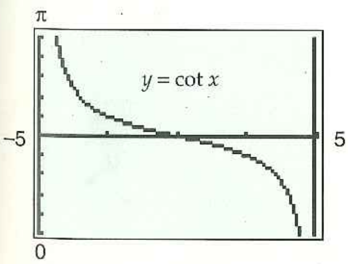


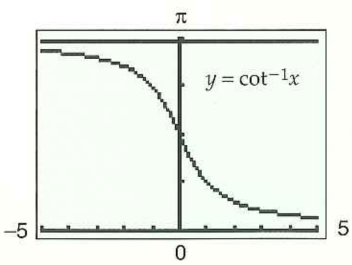
The graph of $y = \csc x$ is one-to-one and has an inverse.



54.



The graph of $y = \cot x$ is one-to-one and has an inverse.



55. *See p. 538.*

Section 8.2 **Inverse Trigonometric Functions** 537

5. a. Let $\cos u = w$ with $0 \le u \le \pi$. Then $u = \cos^{-1} w$, and $\cos^{-1}(\cos u) = \cos^{-1} w = u$. Let $u = \cos^{-1} v$. Then $\cos u = v$, and $\cos(\cos^{-1} v) = \cos u = v$.

b. Let $\tan u = w$ with $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$. Then $u = \tan^{-1} w$, and $\tan^{-1}(\tan u) = \tan^{-1} w = u$. Let $u = \tan^{-1} v$. Then $\tan u = v$, and $\tan(\tan^{-1} v) = \tan u = v$.

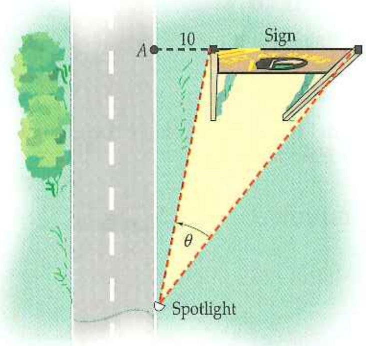
ection

8.3 Algebraic Solutions of rigonometric Equations

eaching Notes

oint out that because trigonometric nctions are periodic functions, there n be an infinite number of solutions trigonometric equations. In fact, ere are an infinite number of lutions to the **basic equations** own here.

56. *Critical Thinking* A 15-foot-wide highway sign is placed 10 feet from a road, perpendicular to the road. A spotlight at the edge of the road is aimed at the sign, as shown in the figure below.



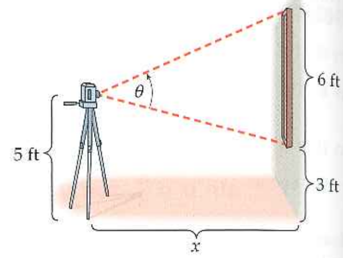
a. Express θ as a function of the distance x from point A to the spotlight.

b. How far from point A should the spotlight be placed so that the angle θ is as large as possible?

a. $\theta = \tan^{-1}\left(\frac{25}{x}\right) - \tan^{-1}\left(\frac{10}{x}\right)$

b. $x \approx 15.8$ **feet**

57. *Critical Thinking* A camera on a 5-foot-high tripod is placed in front of a 6-foot-high picture that is mounted 3 feet above the floor, as shown in figure below.



a. Express angle θ as a function of the distance x from the camera to the wall.

b. The photographer wants to use a particular lens, for which $\theta = 36°\left(\frac{\pi}{5}\text{ radians}\right)$. How far should she place the camera from the wall to be sure the entire picture will show in the photograph?

a. $\theta = \tan^{-1}\left(\frac{4}{x}\right) + \tan^{-1}\left(\frac{2}{x}\right)$

b. $x \approx 9.13$ **feet**

8.3 Algebraic Solutions of Trigonometric Equations

Objective

- Solve trigonometric equations algebraically

Trigonometric equations were solved graphically in Section 8.1. In this section you will learn how to use algebra with inverse trigonometric functions and identities to solve trigonometric equations.

Recall from Section 8.1 that equations such as

$$\sin x = -0.75, \quad \cos x = 0.6, \quad \text{and} \quad \tan x = 3$$

are called **basic equations.** Algebraic solution methods for basic equations are illustrated in Examples 1 through 3.

Example 1 Solving Basic Cosine Equations

Solve $\cos x = 0.6$.