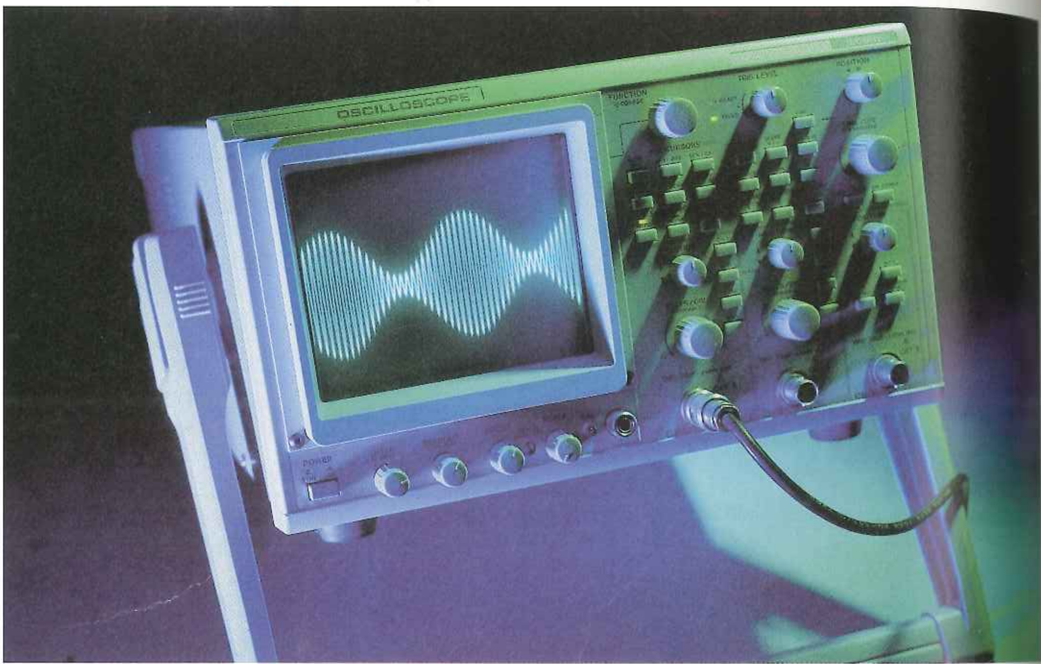
CHAPTER 7

Trigonometric Graphs

n this chapter, students will study raphs of the six trigonometric unctions. They will apply prior nowledge of reflections, stretches, ompressions, shifts, and symmetry o the trigonometric functions. Some ew vocabulary terms associated with pplying these concepts are: periodic raphs, amplitude, and phase shifts. n the Excursion, students will learn bout sinusoidal graphs.

CHAPTER

7 Trigonometric Graphs



Stay tuned for more!

Radio stations transmit by sending out a signal in the form of an electromagnetic wave that can be described by a trigonometric function. The shape of this signal is modified by the sounds being transmitted. AM radio signals are modified by varying the "height," or amplitude, of the waves, whereas FM signals are modified by varying the frequency of the waves. The signal displayed in the photo is from an AM radio station found at 900 on the broadcast dial. See Exercise 65 of Section 7.3.

Chapter Outline

Interdependence of Sections

7.1 ⟶ 7.2
7.1 ⟶ 7.3 ⟶ 7.4

Graphs of trigonometric functions often make it very easy to see the essential properties of these functions, particularly the fact that they repeat their values at regular intervals. Because of the repeating, or periodic, nature of trigonometric functions, they are used to model a variety of phenomena that involve cyclic behavior, such as sound waves, electron orbitals, planetary orbits, radio transmissions, vibrating strings, pendulums, and many more.

7.1 Graphs of the Sine, Cosine, and Tangent Functions

Objectives

- Graph the sine, cosine, and tangent functions
- State all values in the domain of a basic trigonometric function that correspond to a given value of the range
- Graph transformations of the sine, cosine, and tangent graphs

Although a graphing calculator will quickly sketch the graphs of the sine, cosine, and tangent functions, it will not give you much insight into why these graphs have the shapes they do and why these shapes are important. So the emphasis in this section is the connection between the functions' definitions and their graphs.

Using radians and the unit circle, you learned in Chapter 6 that trigonometric functions can be defined as functions of real numbers. Using this definition, you will see that the graphs of trigonometric functions are directly related to angles in the unit circle.

Graph of the Sine Function

Consider an angle of t radians in standard position. Let P be the point where the terminal side of the angle meets the unit circle. Then the y-coordinate of P is the number $\sin t$. As t increases, the graph of $f(t) = \sin t$ can be sketched from the corresponding y-coordinates of P.

473

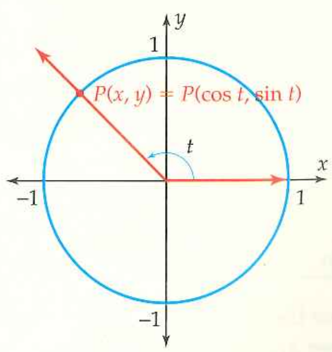
Section

7.1 Graphs of the Sine, Cosine, and Tangent Functions

Teaching Notes

Remind students that the trigonometric functions are defined using t, x, and y. Recall that two consequences of the definitions are:

- For each angle of t radians, the point $P(x, y)$ on the unit circle is $P(\cos t, \sin t)$.
- The tangent ratio is $\tan t = \frac{\sin t}{\cos t}$.



eaching Notes

he full-page diagram illustrates how
o use the y-coordinates on the unit
rcle to sketch one period of the
ne function. To extend the sine
raph to the right, beyond $t = 2\pi$,
t P continue its counterclockwise
ath, retracing around the unit circle.
o extend the graph left of the
axis, let P move clockwise around
e unit circle from 0 to $-\frac{\pi}{2}$, from $-\frac{\pi}{2}$
$-\pi$, and so on. The result is the full
raph of the Sine Function on
ge 475.

sk students to name the points on
e sine graph that correspond to
ese points on the unit circle:
, 0) (0, 1) (−1, 0) (0, −1)
, 0) $\left(\frac{\pi}{2}, 1\right)$ $(\pi, 0)$ $\left(\frac{3\pi}{2}, -1\right)$

sk students to name a local maximum
oint and a local minimum point of the
ne function from the list above.

cal maximum point: $\left(\frac{\pi}{2}, 1\right)$

cal minimum point: $\left(\frac{3\pi}{2}, -1\right)$

om the same list, ask students to
me the x-intercepts of the sine
aph.

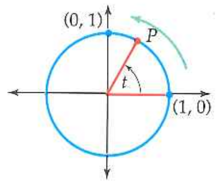
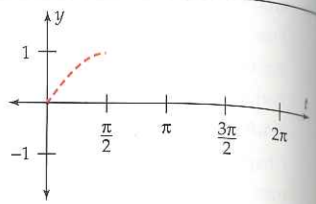
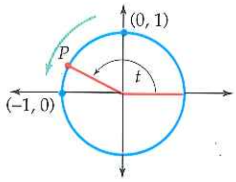
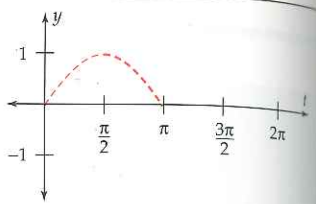
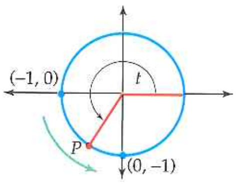
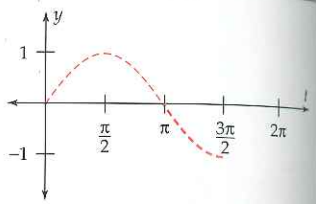
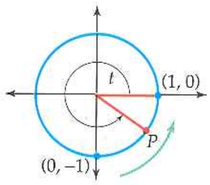
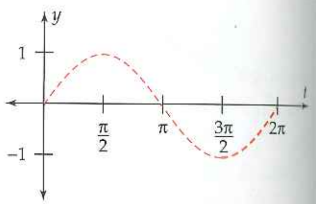
0) and $(\pi, 0)$

courage students to remember the
ints discussed above as reference
ints.

e unit circle can be used to
termine reference points such as
ose discussed above and intervals
which the sine curve is increasing
d decreasing. However, it is not as
eful in determining concavity. To
a more precise sketch, you can
ve students use calculators to
mplete the table below.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
in t	0	0.5	0.707	0.866	1

ve them plot these points with
per and pencil to see that the sine
rve is increasing and concave down
the interval $\left(0, \frac{\pi}{2}\right)$. This activity
ld be extended to verify the
ncavity on other intervals.

Change in t	Movement of point P	$\sin t$ (y-coordinate of P)	Corresponding graph
from 0 to $\frac{\pi}{2}$	from (1, 0) to (0, 1) 	increases from 0 to 1	
from $\frac{\pi}{2}$ to π	from (0, 1) to (−1, 0) 	decreases from 1 to 0	
from π to $\frac{3\pi}{2}$	from (−1, 0) to (0, −1) 	decreases from 0 to −1	
from $\frac{3\pi}{2}$ to 2π	from (0, −1) to (1, 0) 	increases from −1 to 0	

CAUTION

Throughout this chapter, the independent variable used for trigonometric functions will be t to avoid any confusion with the x and y that are part of the definition of these functions. However, using a graphing calculator in function mode, you must enter x as the independent variable.

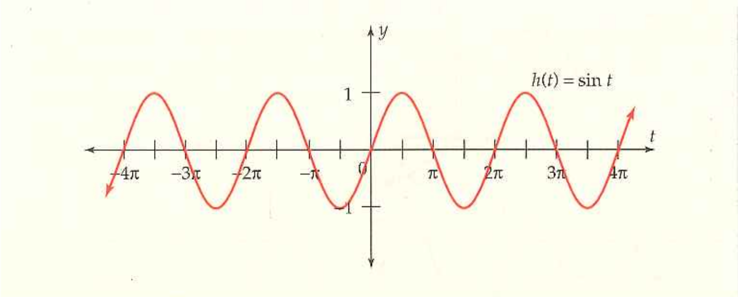
Your graphing calculator can provide a dynamic view of the graph of the sine function and its relationship to points on the unit circle.

> **Graphing Exploration**
>
> With your graphing calculator in parametric mode, set the viewing window as shown below, with a t-step of 0.1.
>
> $$0 \le t \le 2\pi \qquad -\frac{\pi}{3} \le x \le 2\pi \qquad -2.5 \le y \le 2.5$$
>
> On the same screen, graph the two functions given below.
>
> $$X_1 = \cos t, Y_1 = \sin t \qquad X_2 = t, Y_2 = \sin t$$
>
> Use the trace feature to move the cursor along the first graph, which is the unit circle. Stop at a point, and note the values of t and y.
>
> Use the up or down key to move the cursor to the second graph, which is the graph of the sine function. The value of t will remain the same. What are the x- and y-coordinates of this point?
>
> How does the y-coordinate of the new point compare with the y-coordinate of the original point on the unit circle?

To complete the graph of the sine function, note that as t goes from 2π to 4π, the point P on the unit circle *retraces* the path it took from 0 to 2π, so *the same curve will repeat* on the graph. This repetition occurs each 2π units along the horizontal axis, therefore the sine function has a period of 2π. That is, for any real number t,

$$\mathbf{\sin(t \pm 2\pi) = \sin t.}$$

Graph of the Sine Function



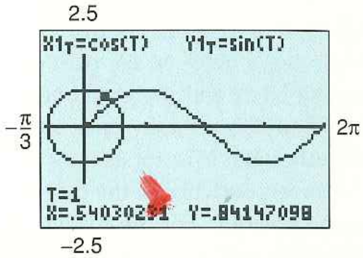
Graph of the Cosine Function

Let P be the point where the terminal side of an angle of t radians in standard position meets the unit circle. Then the x-coordinate of P is the number $\cos t$. To obtain the graph of $f(t) = \cos(t)$, the same process as

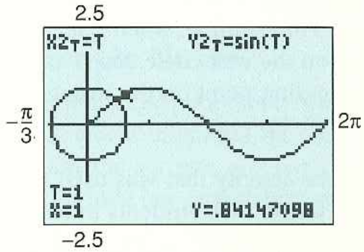
Teaching Notes

Direct students' attention to the **CAUTION** on page 474. Point out that to graph the sine function as shown at the bottom of this page, they would ordinarily use function mode and the independent variable x. However, in the Graphing Exploration at the left, they will be using parametric mode and the independent variable t. Remind students to use radian mode.

Solution to the **Graphing Exploration:**



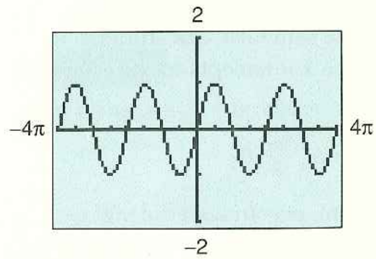
When the cursor is on the unit circle at $t = 1$, $(x, y) = (0.5403, 0.8415)$.



When the cursor is moved to the sine function graph, $(x, y) = (1, 0.8415)$.

The y-coordinate is the same on the sine function graph and the unit circle.

To get a calculator graph of the sine function at the left, students can use $Y_1 = \sin(x)$ in function mode and set the window to $-4\pi \le x \le 4\pi$ and $-2 \le y \le 2$.



To see more or fewer periods, they can modify the x-values in the window.

aching Notes

e graph of the cosine function is tched from the unit circle by a thod similar to that used for the e function. However, the cosine tch could be more difficult for dents. Emphasize the following:

For the sine function, ***y*-values** from he unit circle become y-values on he sine curve.

For the cosine function, ***x*-values** rom the unit circle become -values on the cosine curve.

uitively, it is easier to see the respondence between the path of P the unit circle and the graph of the e function because they "rise and " together. But it is not as easy to the correspondence in the case of cosine function. For the cosine ction, you need to be aware of the *alues* as P travels on the unit circle, l plot points using these values as *alues.* For example, when P is 1, 0) on the unit circle, the responding point on the cosine ve is (0, 1).

peat the activity that was used for sine graph. Ask students to name points on the cosine graph that respond to these points on the unit cle:

0) (0, 1) (−1, 0) (0, −1)

1) $\left(\frac{\pi}{2}, 0\right)$ $(\pi, -1)$ $\left(\frac{3\pi}{2}, 0\right)$

k students to name a local maximum nt and a local minimum point of the ine function from the list above.

al maximum point: (0, 1)

al minimum point: $(\pi, -1)$

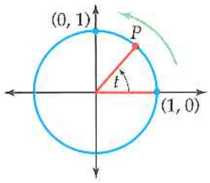
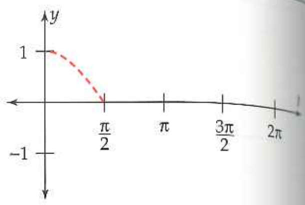
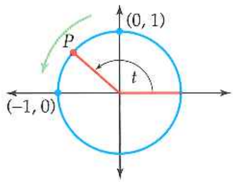
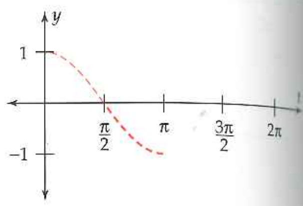
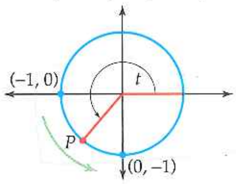
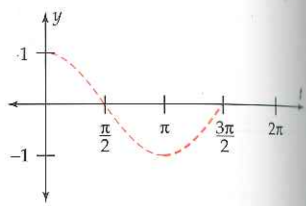
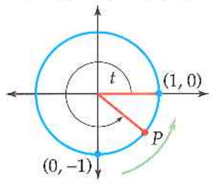
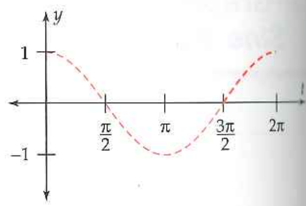
m the same list, ask students to ne the x-intercepts of the cosine ph.

0) and $\left(\frac{3\pi}{2}, 0\right)$

before, encourage students to nember the points discussed above reference points.

u can have students make a table values and plot points to verify cavity as done earlier with the e function.

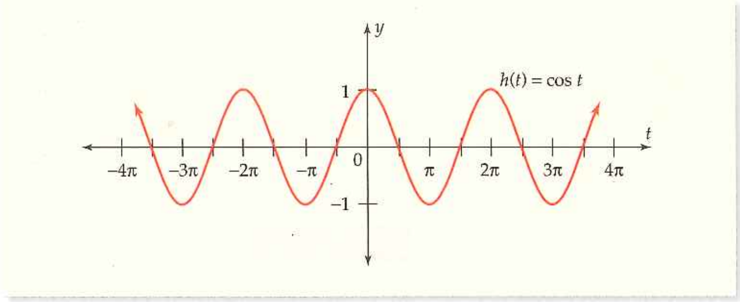
that used for the sine function is followed, except the x-coordinate is observed. The following chart illustrates the graph of the cosine function.

Change in t	Movement of point P	$\cos t$ (x-coordinate of P)	Corresponding graph
from 0 to $\frac{\pi}{2}$	from (1, 0) to (0, 1) 	decreases from 1 to 0	
from $\frac{\pi}{2}$ to π	from (0, 1) to (−1, 0) 	decreases from 0 to −1	
from π to $\frac{3\pi}{2}$	from (−1, 0) to (0, −1) 	increases from −1 to 0	
from $\frac{3\pi}{2}$ to 2π	from (0, −1) to (1, 0) 	increases from 0 to 1	

As the value of t increases, the point P on the unit circle *retraces* its path along the unit circle, so the graph of $f(t) = \cos(t)$ repeats the same curve at intervals of length 2π. Because the cosine function also has a period of 2π, for any number t,

$$\cos(t \pm 2\pi) = \cos t.$$

Graph of the Cosine Function



The graphs of the sine and cosine functions visually illustrate two basic facts about these functions. Because the graphs extend infinitely to the right and to the left,

the domain of the sine and cosine functions is the set of all real numbers.

Also, the y-coordinate of every point on these graphs lies between -1 and 1 (inclusive), so that

the range of the sine and cosine functions is the interval $[-1, 1]$.

You can use the period of the function to state all values of t for which $\sin t$ or $\cos t$ is a given number, as shown in Examples 1 and 2.

Example 1 Finding All *t*-values

State all values of t for which $\sin t$ is -1.

Solution

The sine function oscillates between -1 and 1 and has a period of 2π (i.e., it repeats the pattern every 2π units on the horizontal axis), so there are an *infinite* number of t-values for which $\sin t$ is -1. These points occur every 2π units on the horizontal axis, and a few are highlighted in red on the graph $y = \sin t$ shown in Figure 7.1.1.

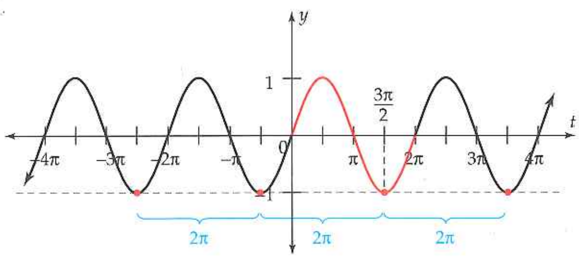
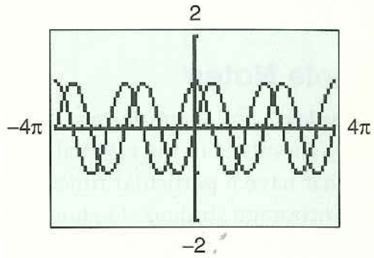


Figure 7.1-1

Teaching Notes

Students can easily see on pages 475 and 477 that the *sine and cosine graphs have identical shapes*. However, it is important for them to realize that the *graphs are not identical* (do not coincide). Have them graph both functions on the same screen using $Y_1 = \sin(x)$ and $Y_2 = \cos(x)$. The window used below is $-4\pi \le x \le 4\pi$ and $-2 \le y \le 2$.



Ask what type of transformation could be applied to one of the graphs to get the other. horizontal shift

Example Notes

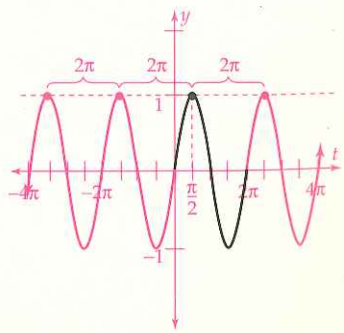
Identifying the period of a function (page 457) is important throughout this section. On a graph, any pair of points that include one complete pattern, or cycle, between them can be used to find the period. The period is the horizontal distance between that pair of points. In **Example 1,** Figure 7.1-1, the period of the sine function is 2π, shown two different ways:

- the red portion of the curve on the interval $[0, 2\pi)$
- successive pairs of local minimum points shown as red dots

ADDITIONAL EXAMPLES

Example 1

State all values of t for which $\sin t$ is 1.



$t = \frac{\pi}{2} + 2k\pi$, where k is any integer

On the interval $[0, 2\pi)$, highlighted in red on the graph above, the graph of $y = \sin t$ has only one point, $\left(\frac{3\pi}{2}, -1\right)$, at which the y-coordinate is -1. Therefore, all values of t for which $\sin t$ is -1 can be expressed as $t = \frac{3\pi}{2} + 2k\pi$, where k is any integer.

Example 2 Finding All *t*-values

State all values of t for which $\cos t$ is $\frac{1}{2}$.

Solution

The cosine function repeats its pattern of y-values at intervals of 2π, so there are an *infinite* number of t-values for which $\cos\ t$ is $\frac{1}{2}$. The graph of $y = \cos t$ shown in Figure 7.1-2 highlights a few points with a y-coordinate of $\frac{1}{2}$.

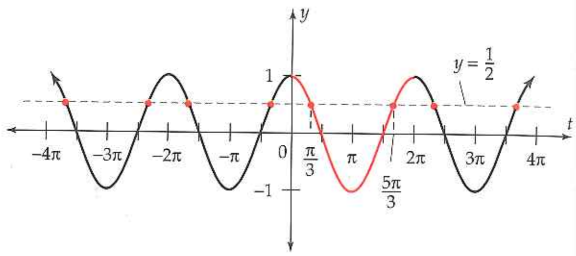


Figure 7.1-2

On the interval $[0, 2\pi)$, highlighted in red on the graph above, the graph of $y = \cos t$ has two points, $\left(\frac{\pi}{3}, \frac{1}{2}\right)$ and $\left(\frac{5\pi}{3}, \frac{1}{2}\right)$, at which the y-coordinate is $\frac{1}{2}$. Therefore, all values of t for which $\cos t$ is $\frac{1}{2}$ can be expressed as $t = \frac{\pi}{3} + 2k\pi$ or $\frac{5\pi}{3} + 2k\pi$, where k is any integer.

Graph of the Tangent Function

To determine the shape of the graph of $f(t) = \tan t$, a connection between the tangent function and slope can be used. As shown in Figure 7.1-3, the point P where the terminal side of an angle of t radians in standard position meets the unit circle has coordinates $(\cos t, \sin t)$. This point and the point $(0, 0)$ can be used to compute the *slope* of the line containing the terminal side.

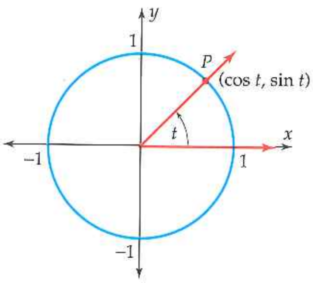


Figure 7.1-3

Teaching Notes

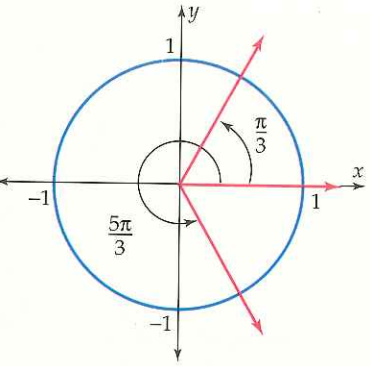
The interval $[0, 2\pi)$ is used to identify the period in Figure 7.1-1, page 477, and in Figure 7.1-2 on this page. Students might wonder why a half-open interval is used. Explain that successive half-open intervals can be "strung together" without overlap to represent the entire domain of the function. The domain consists of these intervals:

$\ldots, [-2\pi, 0), [0, 2\pi), [2\pi, 4\pi), \ldots$

Example Notes

In Examples 1 and 2, graphs are used to identify t-values in the interval $[0, 2\pi)$ that have a particular function value. Encourage students to show the t-values as angles in standard position.

For Example 2, $\frac{\pi}{3}$ and $\frac{5\pi}{3}$ are shown below in standard position. They both have the same cosine value, $\frac{1}{2}$. There are no other angles in the interval $[0, 2\pi)$ that have the cosine value $\frac{1}{2}$.

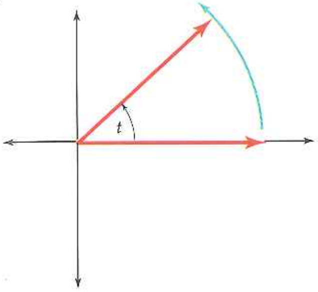
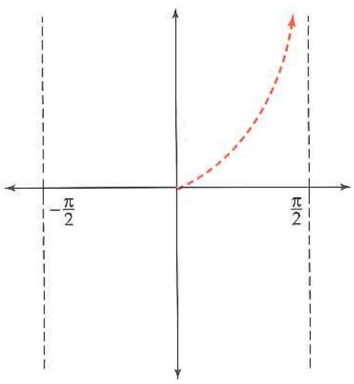
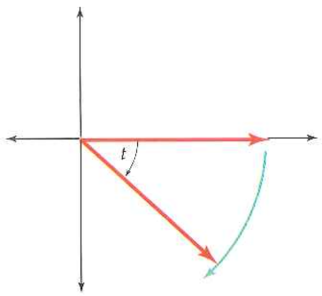
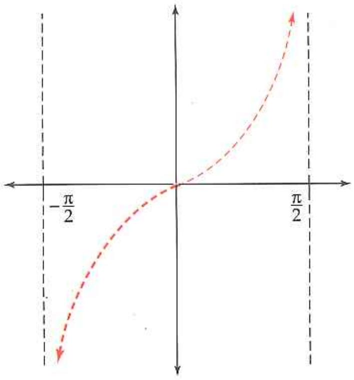


COMMON ERROR ALERT

In **Example 2,** students might identify only one t-value in the interval $[0, 2\pi)$, and then give an incomplete answer. For example, they might identify only $\frac{\pi}{3}$, and give $t = \frac{\pi}{3} + 2k\pi$ as the answer. Remind them to look for *every* point in the interval $[0, 2\pi)$ that has a cosine of $\frac{1}{2}$.

$$\text{slope} = \frac{\sin t - 0}{\cos t - 0} = \frac{\sin t}{\cos t} = \tan t$$

The graph of $f(t) = \tan t$ can be sketched by noting the slope of the terminal side of an angle of t radians, as t takes different values.

Change in t	Movement of terminal side	tan t (terminal side slope)	Corresponding graph
from 0 to $\frac{\pi}{2}$	from horizontal upward toward vertical 	increases from 0 in the positive direction	
from 0 to $-\frac{\pi}{2}$	from horizontal downward toward vertical 	decreases from 0 in the negative direction	

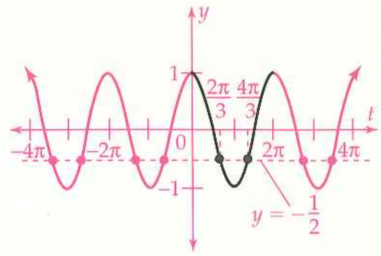
When $t = \pm\frac{\pi}{2}$, the terminal side of the angle is vertical, so its slope is not defined. The graph of the tangent function has vertical asymptotes at the values of t for which the function is undefined.

To complete the graph of the tangent function, note that as t goes from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, the terminal side goes from almost vertical with negative slope to almost vertical with positive slope, exactly as it does from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. So the graph repeats this pattern at intervals of length π.

ADDITIONAL EXAMPLES

Example 2

State all values of t for which $\cos t$ is $-\frac{1}{2}$.



$t = \frac{2\pi}{3} + 2k\pi$ or $t = \frac{4\pi}{3} + 2k\pi$, where k is any integer

Teaching Notes

For the **Graph of the Tangent Function,** review the slope formula and apply it to the points (0, 0) and ($\cos t$, $\sin t$) in Figure 7.1-3:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{\sin t - 0}{\cos t - 0} = \frac{\sin t}{\cos t} = \tan t$$

Remind students that $\frac{0}{x} = 0$ (if $x \neq 0$) and $\frac{x}{0}$ is undefined. Ask what occurs in the graph at t-values for which the function is undefined. vertical asymptotes

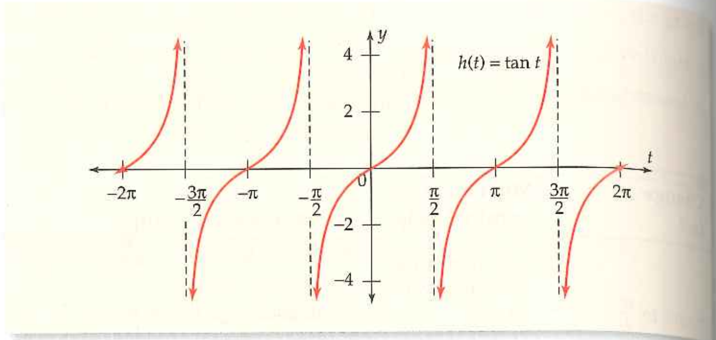
Students can use calculators to complete the table below and then plot points on paper to verify concavity, as done earlier with the sine and cosine functions.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
tan t	0	0.577	1	1.73	undefined

As t approaches $\frac{\pi}{2}$, tangent values increase without bound and the graph gets closer to the asymptote at $\frac{\pi}{2}$. To demonstrate this, note that $\frac{\pi}{2} \approx 1.5707963$. Have students use calculators to find the following tangent values:

$\tan 1.5 \approx$ 14.1
$\tan 1.57 \approx$ 1255.8
$\tan 1.5707 \approx$ 10,381.3

Graph of the Tangent Function



Notice that the **domain of the tangent function** is all real numbers except *odd* multiples of $\frac{\pi}{2}$. The **range of the tangent function** is all real numbers. Because the tangent function has a period of π, for any number t in its domain,

$$\tan(t \pm \pi) = \tan t.$$

Example 3 Finding All t-values

State all values of t for which $\tan t$ is -1.

Solution

The tangent function repeats its pattern of y-values at intervals of π, so there are an *infinite* number t-values for which $\tan t$ is -1. The graph of $y = \tan t$ shown in Figure 7.1-4 highlights a few points with a y-coordinate of -1.

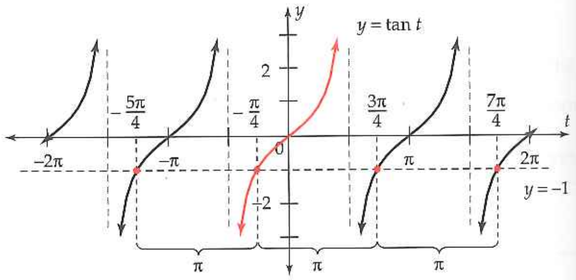


Figure 7.1-4

On the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$, highlighted in red on the graph above, the graph of $y = \tan t$ has only one point, $\left(-\frac{\pi}{4}, -1\right)$, at which the y-coordinate is

Technology Tip

Most calculators have a window setting that automatically rescales the horizontal axis in fractional units of π when in radian mode. On TI models, select ZTRIG in the ZOOM menu, and on Casio, select F3 (V-Window) then F2 (TRIG) from GRAPH mode.

COMMON ERROR ALERT

tudents might think the tangent unction has the set of all real umbers as its domain, or 2π as s period, using the related facts rom the sine and cosine functions. ncourage students to use the graph f the tangent function to help them emember the correct domain and eriod.

eaching Notes

or **Basic Transformations of Sine, osine, and Tangent,** students vill need to review graphical ransformations in Section 3.4, ages 172–181. The Example Notes elow specify the particular types of ransformations needed in each case.

xample Notes

xample 4 is a vertical stretch. It uses he following concept from page 179: or a positive number c,

- If $c > 1$, the graph of $g(x) = c \cdot f(x)$ is the graph of f stretched vertically, away from the x-axis, by a factor of c.
- If $c < 1$, the graph of $g(x) = c \cdot f(x)$ is the graph of f compressed vertically, toward the x-axis, by a factor of c.

xample 5 is a reflection across the -axis and a vertical stretch. The eflection uses this concept from age 177:

- The graph of $g(x) = -f(x)$ is the graph of f reflected across the x-axis.
- The graph of $g(x) = f(-x)$ is the graph of f reflected across the y-axis.

xample 6 is a vertical shift. It uses his concept from page 174: or a positive number c,

- The graph of $g(x) = f(x) + c$ is the graph of f shifted upward c units.
- The graph of $g(x) = f(x) - c$ is the graph of f shifted downward c units.

-1. Therefore, all values of t for which $\tan t$ is -1 can be expressed as $t = -\frac{\pi}{4} + k\pi$, where k is any integer.

Basic Transformations of Sine, Cosine, and Tangent

The graphical transformations (such as shifting and stretching) that were considered in Section 3.4 also apply to trigonometric graphs.

Example 4 Vertical Stretch

List the transformation needed to change the graph of $f(t) = \cos t$ into the graph of $h(t) = 4\cos t$. Graph both equations on the same screen.

Solution

Because $h(t) = 4 \cdot f(t)$, the graph of h is the graph of f after a vertical stretch by a factor of 4. Both graphs are identified in Figure 7.1-5.

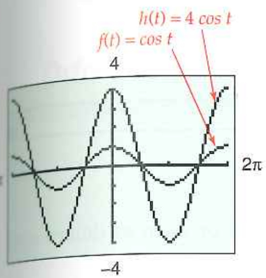


Figure 7.1-5

Example 5 Reflection and Vertical Stretch

Graph $g(t) = -\frac{1}{2}\sin t$ on the interval $[-2\pi, 2\pi]$.

Solution

The graph of g is the graph of $f(t) = \sin t$ reflected across the x-axis and compressed vertically by a factor of $\frac{1}{2}$, as shown in Figure 7.1-6.

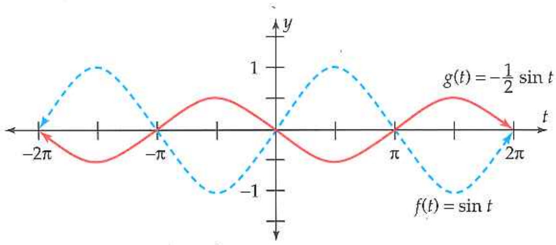


Figure 7.1-6

Example 6 Vertical Shift

Graph $h(t) = \tan t + 5$ on the interval $[-3\pi, 3\pi]$.

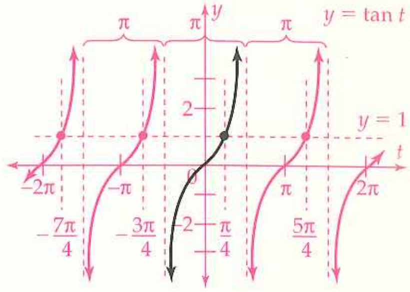
Solution

The graph of h is the graph of $f(t) = \tan t$ shifted up 5 units.

ADDITIONAL EXAMPLES

Example 3

State all values of t for which $\tan t$ is 1.

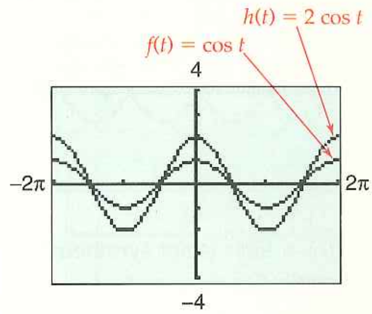


$t = \frac{\pi}{4} + k\pi$, where k is any integer

Example 4

List the transformation needed to change the graph of $f(t) = \cos t$ into the graph of $h(t) = 2\cos t$. Graph both equations on the same screen.

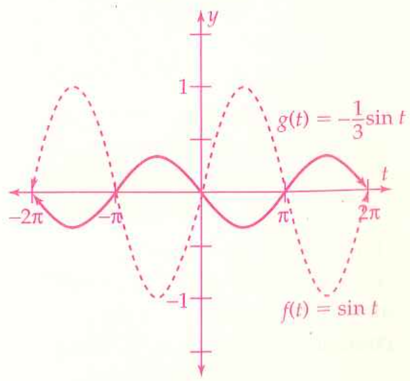
The graph of h is the graph of $f(t) = \cos t$ after a vertical stretch by a factor of 2.



Example 5

Graph $g(t) = -\frac{1}{3}\sin t$ on the interval $[-2\pi, 2\pi]$.

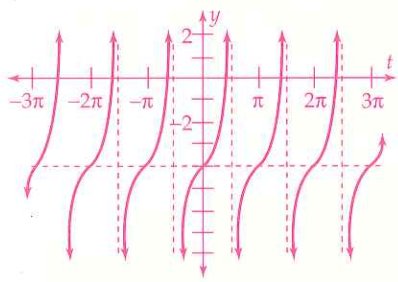
The graph of g is the graph of f reflected across the x-axis and compressed vertically by a factor of $\frac{1}{3}$.



Example 6

Graph $h(t) = \tan t - 4$ on the interval $[-3\pi, 3\pi]$.

The graph of h is the graph of $f(t) = \tan t$ shifted down 4 units.



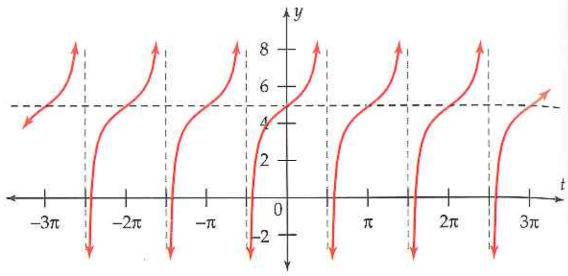


Figure 7.1-7

Even and Odd Functions

Trigonometric functions can be classified as odd or even as determined by their symmetry.

> **NOTE** For a complete discussion of symmetry and odd and even functions, see *Excursion* 3.4.A.

Even Functions

A graph is *symmetric with respect to the y-axis* if the part of the graph on the right side of the y-axis is the mirror image of the part on the left side of the y-axis.

Graphing Exploration

1. For each pair of functions f and g below, answer the following questions.

   $f(t) = \cos t$ and $g(t) = \cos(-t)$
   $f(t) = \sin t$ and $g(t) = \sin(-t)$
   $f(t) = \tan t$ and $g(t) = \tan(-t)$

   - Is f symmetric with respect to the y-axis?
   - Does the graph of g appear to coincide with the graph of f?

2. If a graph is symmetric with respect to the y-axis, describe the graph after a reflection across the y-axis.

A function f whose graph is symmetric with respect to the y-axis is called an **even function.**

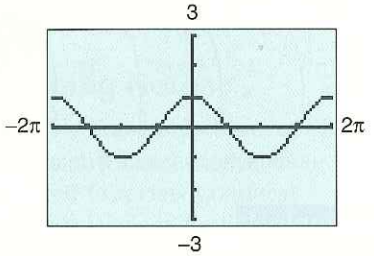
Even Function

> **A function f is *even* if**
>
> **$f(-x) = f(x)$ for every x in the domain of f.**
>
> **The graph of an even function is symmetric with respect to the y-axis.**

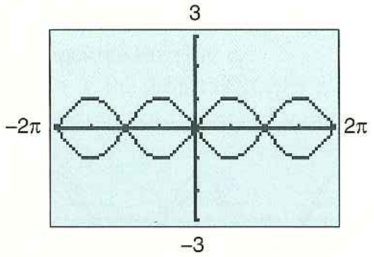
eaching Notes

or the **Graphing Exploration,** udents should use either a ZOOM RIG feature or function mode, radian ode, and windows sufficient to ow complete graphs. The windows sed below are $-2\pi \le x \le 2\pi$ and $3 \le y \le 3$.

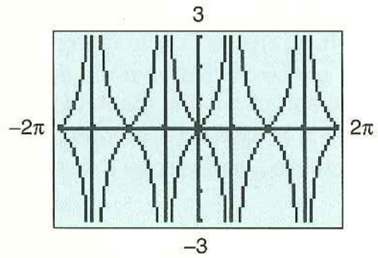
olution to the **Graphing Exploration:**



Yes, $f(t) = \cos t$ is symmetric with respect to the y-axis.
Yes, the graph of $g(t) = \cos(-t)$ appears to coincide with the graph of $f(t) = \cos t$.



No, $f(t) = \sin t$ is not symmetric with respect to the y-axis.
No, the graph of $g(t) = \sin(-t)$ does not coincide with the graph of $f(t) = \sin t$.



No, $f(t) = \tan t$ is not symmetric with respect to the y-axis.
No, the graph of $g(t) = \tan(-t)$ does not coincide with the graph of $f(t) = \tan t$.

. If a graph is symmetric with respect to the y-axis, the image after reflecting it across the y-axis coincides with the original graph.

Math Background

The even and odd functions in Excursion 3.4.A were polynomials such as $y = x^4 + x^2$ and $y = x^3 + 2x$. For polynomial functions, the classifications even and odd are natural and intuitive because all terms of an even function have even degree and all terms of an odd function have odd degree.

Here, we classify sine, cosine, and tangent by using the same definitions stated in the Excursion. The definitions use symmetry with respect to an axis; they are not dependent on degree.

For example, $f(t) = \cos t$ is an even function because

$$\cos(-t) = \cos t \text{ for every } t \text{ in the domain of } f(t) = \cos t.$$

Odd Functions

If a graph is *symmetric with respect to the origin*, then whenever (x, y) is on the graph, $(-x, -y)$ is also on the graph. A function f whose graph is symmetric with respect to the origin is called an **odd function.**

Odd Function

> **A function f is *odd* if**
>
> $$f(-x) = -f(x) \text{ for every } x \text{ in the domain of } f.$$
>
> **The graph of an odd function is symmetric with respect to the origin.**

For example, $f(t) = \sin t$ and $g(t) = \tan t$ are odd functions because

$$\sin(-t) = -\sin t \text{ for every } t \text{ in the domain of } f(t) = \sin t$$
$$\tan(-t) = -\tan t \text{ for every } t \text{ in the domain of } g(t) = \tan t.$$

Summary of the Properties of Sine, Cosine, and Tangent Functions

Function	Symbol	Domain	Range	Period	Even/Odd
sine	$f(t) = \sin t$	all real numbers	all real numbers from −1 to 1, inclusive	2π	odd
cosine	$f(t) = \cos t$	all real numbers	all real numbers from −1 to 1, inclusive	2π	even
tangent	$f(t) = \tan t$	all real numbers except odd multiples of $\frac{\pi}{2}$	all real numbers	π	odd

Exercises 7.1

In Exercises 1–6, graph each function on the given interval.

1. $f(t) = \sin t; [2\pi, 6\pi]$
2. $g(t) = \cos t; [\pi, 3\pi]$
3. $h(t) = \tan t; [\pi, 2\pi]$
4. $f(t) = \sin t; [-5\pi, -3\pi]$
5. $g(t) = \cos t; \left[\frac{7\pi}{6}, \frac{7\pi}{2}\right]$
6. $h(t) = \tan t; \left[\frac{5\pi}{3}, 3\pi\right]$
7. For what values of t on the interval $[-2\pi, 2\pi]$ is $\sin t = 1$? $-\frac{3\pi}{2}, \frac{\pi}{2}$
8. For what values of t on the interval $[-2\pi, 2\pi]$ is $\cos t = 0$? $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$
9. What is the maximum value of $g(t) = \cos t$? 1
10. What is the minimum value of $f(t) = \sin t$? −1

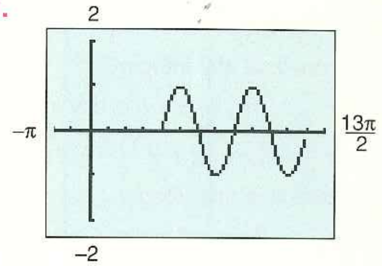
Real-World Application

Graphs of trigonometric functions and their transformations can be used to model natural and man-made phenomena. The graph in Figure 7.1-7 on page 482 is a transformation of the tangent function. A different transformation of the tangent function graph is used in Exercise 61 to model the position of a rotating beam of light. That graph is then used to estimate the position at various times as the beam rotates.

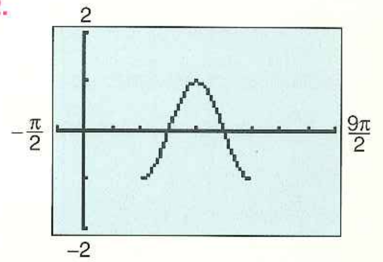
Exercises 7.1

ANSWERS

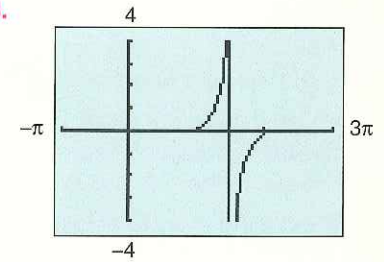
1.



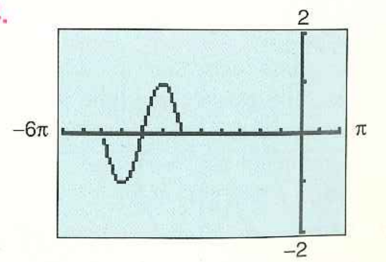
2.



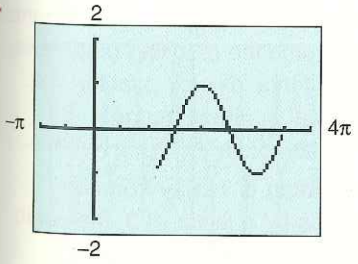
3.



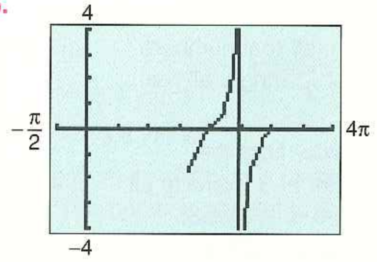
4.



5.



6.



17. $-\frac{7\pi}{4} < t < -\frac{3\pi}{2}$, $-\frac{3\pi}{4} < t < -\frac{\pi}{2}$, $\frac{\pi}{4} < t < \frac{\pi}{2}$, and $\frac{5\pi}{4} < t < \frac{3\pi}{2}$

18. $-\frac{3\pi}{2} < t < \pi$, $-\frac{\pi}{2} < t < 0$, $\frac{\pi}{2} < t < \pi$, and $\frac{3\pi}{2} < t < 2\pi$

19. all values on the interval $[\pi, 2\pi]$ except $\frac{3\pi}{2}$

20. $t = 0 + n\pi$, where n is any integer

21. $t = \frac{\pi}{4} + 2n\pi$ or $t = \frac{3\pi}{4} + 2n\pi$, where n is any integer

22. $t = n\pi$, where n is any integer

23. $t = \frac{2\pi}{3} + 2n\pi$ or $t = \frac{4\pi}{3} + 2n\pi$, where n is any integer

24. $t = \frac{\pi}{4} + n\pi$, where n is any integer

25. $t = \frac{4\pi}{3} + 2n\pi$ or $t = \frac{5\pi}{3} + 2n\pi$, where n is any integer

26. $t = \frac{\pi}{2} + n\pi$, where n is any integer

27. $t = \frac{\pi}{6} + 2n\pi$ or $t = \frac{11\pi}{6} + 2n\pi$, where n is any integer

28. $t = \frac{\pi}{2} + 2n\pi$, where n is any integer

29. $t = \frac{\pi}{6} + 2n\pi$ or $t = \frac{5\pi}{6} + 2n\pi$, where n is any integer

30. $t = \frac{5\pi}{6} + n\pi$, where n is any integer

31. $t = \frac{3\pi}{4} + 2n\pi$ or $t = \frac{5\pi}{4} + 2n\pi$, where n is any integer

32. $t = \pi + 2n\pi$, where n is any integer

33. $t = \frac{\pi}{3} + n\pi$, where n is any integer

34. Shift the graph of f vertically 2 units downward. domain: all real numbers; range: $-3 \le g(t) \le -1$

35. Reflect the graph of f across the horizontal axis. domain: all real numbers; range: $-1 \le g(t) \le 1$

36. Reflect the graph of f across the horizontal axis, then stretch the resulting graph away from the horizontal axis by a factor of 3. domain: all real numbers; range $-3 \le g(t) \le 3$

37. Shift the graph of f vertically 5 units upward. domain: all real numbers except odd multiples of $\frac{\pi}{2}$; range: all real numbers

11. For what values of t on the interval $[-2\pi, 2\pi]$ does the graph of $h(t) = \tan t$ have vertical asymptotes? $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

12. What is the y-intercept of the graph of $f(t) = \sin t$? **0**

13. What is the y-intercept of the graph of $g(t) = \cos t$? **1**

14. What is the y-intercept of the graph of $h(t) = \tan t$? **0**

15. For what values of t on the interval $[-\pi, \pi]$ is $f(t) = \sin t$ increasing? $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

16. For what values of t on the interval $[-3\pi, -\pi]$ is $g(t) = \cos t$ decreasing? $-2\pi \le t \le -\pi$

17. For what values of t on the interval $[-2\pi, 2\pi]$ is $\tan t$ greater than 1?

18. For what values of t on the interval $[-2\pi, 2\pi]$ is $\tan t$ less than 0?

19. For what values of t on the interval $[\pi, 2\pi]$ is $h(t) = \tan t$ increasing?

In Exercises 20–33, find all the exact t-values for which the given statement is true.

20. $\tan t = 0$
21. $\sin t = \frac{\sqrt{2}}{2}$
22. $\sin t = 0$
23. $\cos t = -\frac{1}{2}$
24. $\tan t = 1$
25. $\sin t = -\frac{\sqrt{3}}{2}$
26. $\cos t = 0$
27. $\cos t = \frac{\sqrt{3}}{2}$
28. $\sin t = 1$
29. $\sin t = \frac{1}{2}$
30. $\tan t = -\frac{\sqrt{3}}{3}$
31. $\cos t = -\frac{\sqrt{2}}{2}$
32. $\cos t = -1$
33. $\tan t = \sqrt{3}$

In Exercises 34–43, list the transformations that change the graph of f into the graph of g. State the domain and range of g.

34. $f(t) = \cos t$; $g(t) = \cos t - 2$
35. $f(t) = \cos t$; $g(t) = -\cos t$
36. $f(t) = \sin t$; $g(t) = -3 \sin t$
37. $f(t) = \tan t$; $g(t) = \tan t + 5$
38. $f(t) = \tan t$; $g(t) = -\tan t$
39. $f(t) = \cos t$; $g(t) = 3 \cos t$
40. $f(t) = \sin t$; $g(t) = -2 \sin t$
41. $f(t) = \sin t$; $g(t) = 3 \sin t + 2$
42. $f(t) = \cos t$; $g(t) = 5 \cos t + 3$
43. $f(t) = \sin t$; $g(t) = \sin t + 3$

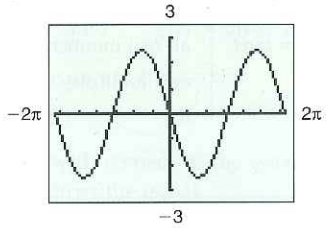
In Exercises 44–48, sketch the graph of each function.

44. $f(t) = -2 \cos t$
45. $f(t) = 5 \sin t + 1$
46. $f(t) = 4 \tan t$
47. $f(t) = -\frac{1}{4} \cos t$
48. $f(t) = 3 \sin t - \frac{1}{2}$

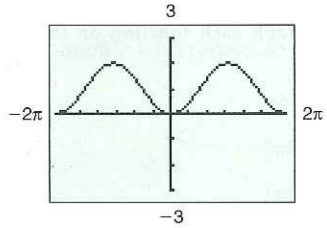
In Exercises 49–54, match a graph to a function. Only one graph is possible for each function.

49. $h(t) = -2 \tan t$ **d**
50. $g(t) = 2.5 \cos t$ **c**
51. $h(t) = -\sin t + 1$ **e**
52. $f(t) = -2.5 \sin t$ **a**
53. $g(t) = 3 \tan t - 1$ **f**
54. $f(t) = -\cos t + 1$ **b**

a.



b.



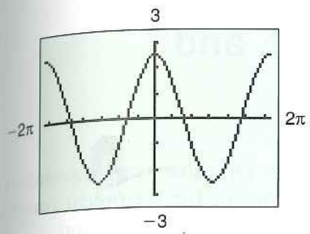
38. Reflect the graph of f across the horizontal axis. domain: all real numbers except odd multiples of $\frac{\pi}{2}$; range: all real numbers

39. Stretch the graph of f away from the horizontal axis by a factor of 3. domain: all real numbers; range: $-3 \le g(t) \le 3$

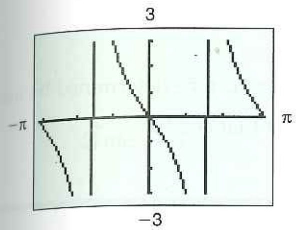
40. Reflect the graph of f across the horizontal axis, then stretch the resulting graph away from the horizontal axis by a factor of 2. domain: all real numbers; range: $-2 \le g(t) \le 2$

41. Stretch the graph of f away from the horizontal axis by a factor of 3, then shift the resulting graph vertically 2 units upward. domain: all real numbers; range: $-1 \le g(t) \le 5$

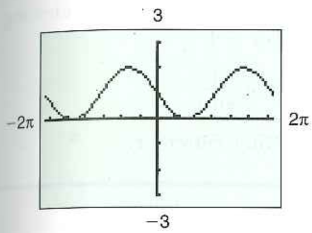
c.



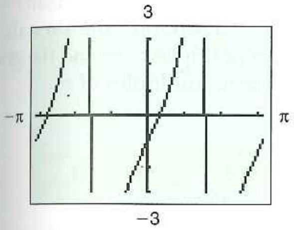
d.



e.



f.



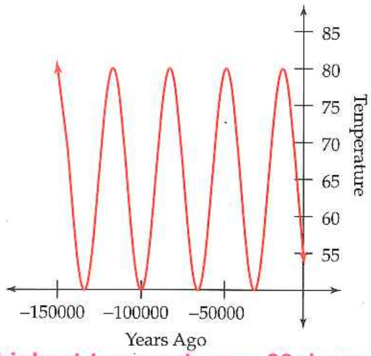
55. Fill the blanks with "even" or "odd" so that the resulting statement is true. Then prove the statement by using an appropriate identity. *Excursion* 3.4.A may be helpful.

a. $f(t) = \sin t$ is an ___ function. **odd; $\sin(-t) = -\sin t$**
b. $g(t) = \cos t$ is an ___ function. **even; $\cos(-t) = \cos t$**
c. $h(t) = \tan t$ is an ___ function. **odd; $\tan(-t) = -\tan t$**
d. $f(t) = t \sin t$ is an ___ function. **even; $\sin(-t) = -\sin t$**
e. $g(t) = t + \tan t$ is an ___ function. **odd; $\tan(-t) = -\tan t$**

In Exercises 56–59, find tan *t*, where the terminal side of an angle of *t* radians lies on the given line.

56. $y = 1.5x$ **1.5**
57. $y = 1.4x$ **1.4**
58. $y = 0.32x$ **0.32**
59. $y = 11x$ **11**

60. Scientists theorize that the average temperature at a specific location fluctuates from cooler to warmer and then to cooler again over a long period of time. The graph shows a theoretical prediction of the average summer temperature for the last 150,000 years for a location in Alaska.



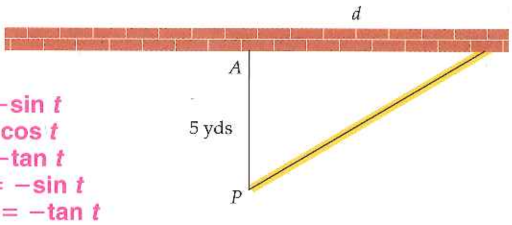
highest temperature = 80; lowest = 50

a. Find the highest and lowest temperature represented.
b. Over what time interval does the temperature repeat the cycle? **about 34,558 years**
c. What is the estimated average summer temperature at the present time? **about 55**

61. A rotating beacon is located at point *P*, 5 yards from a wall. The distance *d*, as measured along the wall, where the light shines is given by

$$d = 5 \tan 2\pi t$$

where *t* is time measured in seconds since the beacon began to rotate. When $t = 0$, the light is aimed at point *A*. When the beacon is aimed to the right of *A*, the distance *d* is positive, and when it is aimed to the left of *A*, the value of *d* is negative.



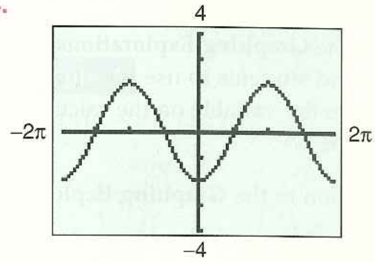
Graph the function and estimate the value of *d* for the following times.

a. $t = 0$
b. $t = 0.5$
c. $t = 0.7$
d. $t = 1.4$
e. Determine the position of the beacon when $t = 0.25$ and discuss the corresponding value of *d* for that value of *t*.

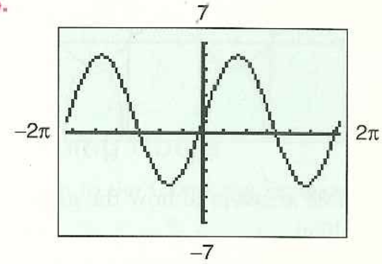
42. Stretch the graph of *f* away from the horizontal axis by a factor of 5, then shift the resulting graph vertically 3 units upward. domain: all real numbers; range: $-2 \le g(t) \le 8$

43. Shift the graph of *f* vertically 3 units upward. domain: all real numbers; range: $2 \le g(t) \le 4$

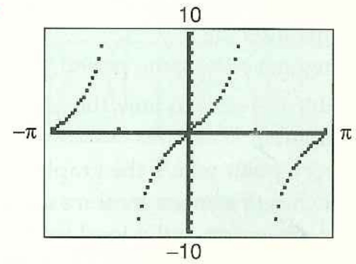
44.



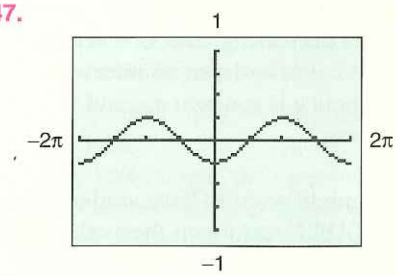
45.



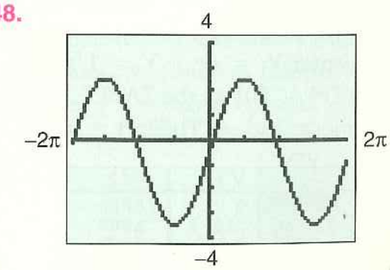
46.



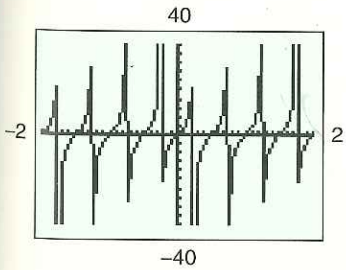
47.



48.



61.



a. 0
b. 0
c. ≈15.4 yards
d. ≈−3.6 yards
e. When $t = 0.25$, *d* is undefined. The beam is parallel to the wall at this time.