Section 6.5 Basic Trigonometric Identities

Teaching Notes

Point out that while using $\sin t^3$ in place of $\sin (t^3)$ is common, the latter notation is less ambiguous. It is clear in the latter notation that one is taking the sine of the quantity t^3, while one might mistake $\sin t^3$ as meaning $(\sin t)^3$.

Encourage students to use grouping symbols with trigonometric functions as most calculators do, illustrated in the margin **NOTE.** By developing the habit of, for example, writing the cosine function as cos ( ) and then filling in the expression, mistakes such as writing $\cos t + 3$ instead of $\cos (t + 3)$ can always be avoided.

In Figure 6.5-2, point out that even though sin (4^3 and sin (4^3) produce the same value on some calculators, not closing the parentheses could lead to problems in more complicated calculations.

Remind students that the notation $\sin^{-1} \theta$ does not indicate the reciprocal function $\frac{1}{\sin \theta}$ (**Technology Tip** second bullet, page 417). To indicate the reciprocal function use $(\sin \theta)^{-1} = \frac{1}{\sin \theta}$.

6.5 Basic Trigonometric Identities

Objectives

- Develop basic trigonometric identities

NOTE Most calculators automatically insert an opening parenthesis when a trigonometric function key is pushed. The display cos(5 + 3 is interpreted as cos(5 + 3). If you want cos 5 + 3, you must insert a parenthesis after the 5: cos(5) + 3.

Technology Tip

Calculators do not use the convention of writing an exponent between the trigonometric function and its argument. In order to obtain $\sin^3 4$, you must enter sin (4)^3.

The algebra of trigonometric functions is just like that of other functions. They may be added, subtracted, composed, etc. However, two notational conventions are normally used with trigonometric functions.

Parentheses can be omitted whenever no confusion can result.

Figure 6.5-1 shows, however, that parentheses *are* needed to distinguish

$$\cos(t + 3) \quad \text{and} \quad \cos t + 3.$$

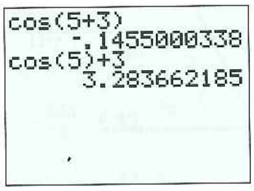


Figure 6.5-1

When dealing with powers of trigonometric functions,

exponents (other than −1) are written between the function symbol and the variable.

For example,

$$(\cos t)^3 \quad \text{is written} \quad \cos^3 t.$$

Furthermore,

$$\sin t^3 \quad \text{means } \sin(t^3) \quad \textit{not } (\sin t)^3 \text{ or } \sin^3 t,$$

as illustrated in Figure 6.5-2.

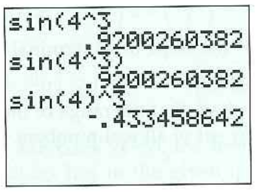


Figure 6.5-2

Identities

Trigonometric functions have numerous relationships that can be expressed as *identities*. An **identity** is an equation that is true for all values of the variables for which every term of the equation is defined. For example,

$$(a + b)^2 = a^2 + 2ab + b^2$$

is an identity because it is true for all possible values of a and b.

The unit circle description of trigonometric functions (see the box on page 446) leads to the following **quotient identities.**

Quotient Identities

$$\tan t = \frac{\sin t}{\cos t} \qquad \cot t = \frac{\cos t}{\sin t}$$

Example 1 Quotient Identities

Simplify the expression below.

$$\tan t \cos t$$

Solution

By the quotient identity,

$$\tan t \cos t = \frac{\sin t}{\cos t}\cos t = \sin t$$

Reciprocal Identities

The **reciprocal identities** follow immediately from the definitions of the trigonometric functions.

Reciprocal Identities

$$\sin t = \frac{1}{\csc t} \qquad \cos t = \frac{1}{\sec t} \qquad \tan t = \frac{1}{\cot t}$$

$$\csc t = \frac{1}{\sin t} \qquad \sec t = \frac{1}{\cos t} \qquad \cot t = \frac{1}{\tan t}$$

CAUTION

An identity may not be true for a value of the variable that makes a term of the equation undefined. For example, if $t = 0$, then $\tan t = 0$ while $\cot t$ is undefined. Thus, $\tan t \neq \frac{1}{\cot t}$ for $t = 0$.

Example 2 Reciprocal Identities

Given that $\sin t = 0.28$ and $\cos t = 0.96$, find $\csc t$ and $\sec t$.

Solution

By the reciprocal identities,

$$\csc t = \frac{1}{\sin t} = \frac{1}{0.28} \approx 3.57 \qquad \sec t = \frac{1}{\cos t} = \frac{1}{0.96} \approx 1.04$$

ADDITIONAL EXAMPLES

Example 1

Simplify the expression $\cot t \sin t$.

$\cot t \sin t = \frac{\cos t}{\sin t} \sin t = \cos t$

Example 2

Given that $\cos t = 0.75$ and $\tan t = 0.88$, find $\sec t$ and $\cot t$.

$\sec t \approx 1.33$, $\cot t \approx 1.14$

Teaching Notes

The **Quotient Identities** were justified using the **Unit Circle Description of Trigonometric Functions** (page 446). An alternate way of justifying the quotient identities is to use the definition of **Trigonometric Functions of a Real Variable** (page 445):

$$\frac{\sin t}{\cos t} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r}\cdot\frac{r}{x} = \frac{y}{x} = \tan t \qquad \frac{\cos t}{\sin t} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{r}\cdot\frac{r}{y} = \frac{x}{y} = \cot t$$

The **Reciprocal Identities** can be proven in a similar fashion.

Math Background

There are two kinds of trigonometric equations: identities and conditional equations. A trigonometric identity is an equation that is valid for all values of the variable for which all terms of the equation are defined, such as $\tan t = \frac{\sin t}{\cos t}$. A conditional equation is valid only for certain values of the variable. For example, the solution of $\sin x = 0$ is $x = \pm k\pi$, where k is an integer.

The **Basic Trigonometric Identities** are discussed in this section and identities are further studied in Chapter 9 **(Trigonometric Identities and Proof).** Conditional equations are dealt with in Chapter 8 **(Solving Trigonometric Equations).**

In the **CAUTION** note, the values for which $\tan t = 0$ are $t = \pm k\pi$, where k is an integer. This is a direct result of the **Periodicity Identities** (page 458).

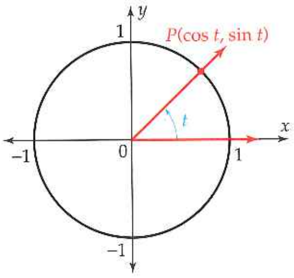


Figure 6.5-3

Pythagorean Identities

For any real number t, the coordinates of the point P where the termin side of an angle of t radians meets the unit circle are $(\cos t, \sin t)$, as show in Figure 6.5-3. Since P is on the unit circle, its coordinates must satisf $x^2 + y^2 = 1$, which is the equation of the unit circle. That is,

$$\cos^2 t + \sin^2 t = 1$$

This identity, which is usually written $\sin^2 t + \cos^2 t = 1$, is called th **Pythagorean identity.** It can be used as follows to derive two other iden- tities, which are also called **Pythagorean identities.**

$$\sin^2 t + \cos^2 t = 1$$

$$\frac{\sin^2 t}{\cos^2 t} + \frac{\cos^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} \qquad \textit{Divide by } \cos^2 t$$

$$\tan^2 t + 1 = \sec^2 t \qquad \textit{Simplify}$$

Similarly, dividing both sides of $\sin^2 t + \cos^2 t = 1$ by $\sin^2 t$ shows that

$$1 + \cot^2 t = \csc^2 t$$

Pythagorean Identities

$$\mathbf{\sin^2 t + \cos^2 t = 1}$$
$$\mathbf{\tan^2 t + 1 = \sec^2 t}$$
$$\mathbf{1 + \cot^2 t = \csc^2 t}$$

In addition to the version shown above, the following forms of the Pythagorean identity are also commonly used.

$$\sin^2 t = 1 - \cos^2 t$$
$$\cos^2 t = 1 - \sin^2 t$$

Example 3 Pythagorean Identities

Simplify the expression below.

$$\tan^2 t \cos^2 t + \cos^2 t$$

Solution

By the quotient and Pythagorean identities,

$$\tan^2 t \cos^2 t + \cos^2 t = \frac{\sin^2 t}{\cos^2 t}\cos^2 t + \cos^2 t = \sin^2 t + \cos^2 t = 1$$

Periodicity Identities

Let t be any real number. Construct two angles in standard position of measure t and $t + 2\pi$ radians, as shown in Figure 6.5-4. Since both of

eaching Notes

he **Pythagorean Identities** are based n the Pythagorean Theorem. They an be proven using the definition of **rigonometric Functions of a Real ariable.**

$$\sin^2 t + \cos^2 t = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{y^2 + x^2}{r^2}$$
$$= \frac{r^2}{r^2} = 1$$

$$\tan^2 t + 1 = \frac{y^2}{x^2} + \frac{x^2}{x^2} = \frac{y^2 + x^2}{x^2}$$
$$= \frac{r^2}{x^2} = \sec^2 t$$

$$1 + \cot^2 t = \frac{y^2}{y^2} + \frac{x^2}{y^2} = \frac{y^2 + x^2}{y^2}$$
$$= \frac{r^2}{y^2} = \csc^2 t$$

xample Notes

n **Example 3,** an alternate solution, y the Pythagorean and reciprocal dentities, is given below.

$$\tan^2 t \cos^2 t + \cos^2 t = \cos^2 t(\tan^2 t + 1)$$
$$= \cos^2 t \sec^2 t$$
$$= \cos^2 t\left(\frac{1}{\cos^2 t}\right)$$
$$= 1$$

DDITIONAL EXAMPLES

Example 3

implify the expression below.

$\cot^2 t \sin^2 t + \sin^2 t$

$$\cot^2 t \sin^2 t + \sin^2 t$$
$$= \frac{\cos^2 t}{\sin^2 t}\sin^2 t + \sin^2 t$$
$$= \cos^2 t + \sin^2 t$$
$$= 1$$

or

$$\cot^2 t \sin^2 t + \sin^2 t = \sin^2 t(\cot^2 t + 1)$$
$$= \sin^2 t \csc^2 t$$
$$= \sin^2 t \frac{1}{\sin^2 t}$$
$$= 1$$

these angles have the same terminal side, the point P where the terminal side intersects the unit circle is the same for both angles.

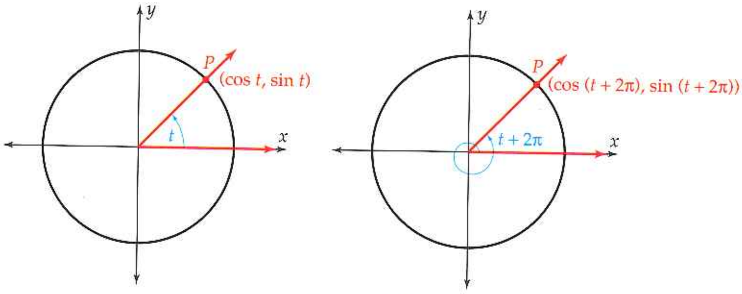


Figure 6.5-4

In both cases, the sine is the y-coordinate of P, so

$$\sin t = \sin(t + 2\pi).$$

In addition, the terminal side of the angle is the same for measures of t, $t \pm 2\pi$, $t \pm 4\pi$, $t \pm 6\pi$, and so on. Thus,

$$\sin t = \sin(t \pm 2\pi) = \sin(t \pm 4\pi) = \sin(t \pm 6\pi) = \ldots$$

Similarly in both cases, the cosine is the x-coordinate of P, so

$$\cos t = \cos(t \pm 2\pi) = \cos(t \pm 4\pi) = \cos(t \pm 6\pi) = \ldots$$

The identities above show that sine and cosine functions repeat their values at regular intervals. Such functions are called *periodic*. A function is said to be **periodic** if there exists some constant k such that

$$f(t) = f(t + k)$$

for every number t in the domain of f. The smallest value of k that has this property is called the **period** of the function f.

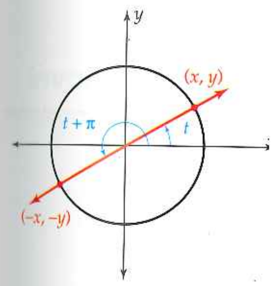


Figure 6.5-5

Since the tangent function is the quotient of the sine and cosine functions, it must also be true that $\tan t = \tan(t + 2\pi)$. However, there is a number smaller than 2π that has this property. Figure 6.5-5 shows the angles t and $t + \pi$. A rotation of π radians is the same as a rotation of 180°, so the image of the point (x, y) is $(-x, -y)$. Thus,

$$\tan(t + \pi) = \frac{-y}{-x} = \frac{y}{x} = \tan t$$

Calculator Exploration

Use your calculator to verify the following:

$$\sin 3 = \sin(3 + 2\pi) = \sin(3 - 4\pi)$$
$$\cos 4 = \cos(4 + 2\pi) = \cos(4 + 6\pi)$$
$$\tan 1 = \tan(1 + \pi) = \tan(1 - 5\pi)$$

Teaching Notes

The derivation, in the text, of the identity $\tan(t + \pi) = \tan t$ uses angles in Quadrant I and Quadrant III. A similar argument can be made for angles in Quadrant II and Quadrant IV. In particular, if t is in Quadrant II, then $t + \pi$ lands in Quadrant IV. In such case, if the intercept of the terminal side of t has coordinates $(-x, y)$, then $\tan t = \frac{y}{-x}$ and $\tan(t + \pi) = \frac{-y}{x}$. Since $\frac{y}{-x} = \frac{-y}{x}$, we can conclude that $\tan(t + \pi) = \tan t$.

Solution to the **Calculator Exploration:**

For $\sin 3 = \sin(3 + 2\pi)$,
$= \sin(3 - 4\pi)$,
enter the expressions below.

```
sin(3)
          .1411200081
sin(3+2π)
          .1411200081
sin(3-4π)
          .1411200081
```

For $\cos 4 = \cos(4 + 2\pi)$
$= \cos(4 + 6\pi)$,
enter the expressions below.

```
cos(4)
         -.6536436209
cos(4+2π)
         -.6536436209
cos(4+6π)
         -.6536436209
```

For $\tan 1 = \tan(1 + \pi)$
$= \tan(1 - 5\pi)$,
enter the expressions below.

```
tan(1)
          1.557407725
tan(1+π)
          1.557407725
tan(1-5π)
          1.557407725
```

Periodicity Identities

The sine and cosine functions are periodic with period 2π. For every real number t,

$$\sin(t \pm 2\pi) = \sin t \qquad \text{and} \qquad \cos(t \pm 2\pi) = \cos t$$

The tangent function is periodic with period π. For every number t in the domain of the tangent function,

$$\tan(t \pm \pi) = \tan t$$

Example 4 Periodicity Identities

Find the exact value of $\sin \frac{13\pi}{6}$.

Solution

By the periodicity identity for sine,

$$\sin \frac{13\pi}{6} = \sin\left(\frac{\pi}{6} + \frac{12\pi}{6}\right) = \sin\left(\frac{\pi}{6} + 2\pi\right) = \sin \frac{\pi}{6} = \frac{1}{2}$$

Negative Angle Identities

Let t be any real number and construct two angles in standard position of measure t and $-t$ radians, as shown in Figure 6.5-6.

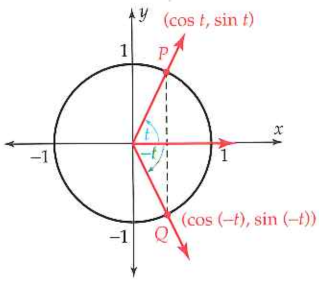


Figure 6.5-6

Since the point Q is the reflection of the point P across the x-axis, the x-coordinates of P and Q are the same, and the y-coordinates are opposites of each other. Thus,

$$\cos t = \cos(-t) \qquad \text{and} \qquad \sin t = -\sin(-t)$$

Also,

$$\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin t}{\cos t} = -\frac{\sin t}{\cos t} = -\tan t$$

Teaching Notes

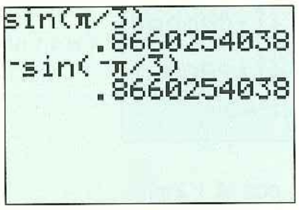
In **Negative Angle Identities,** emphasize the role that the symmetry of the circle plays in the proof.

Remind the students of the definition of *even* and *odd* functions:

- f is *even* if $f(t) = f(-t)$, for all t in the domain of f
- f is *odd* if $-f(t) = f(-t)$, for all t in the domain of f.

So, cosine is an even function and sine and tangent are odd functions. These identities are sometimes called the Even-Odd Identities.

To verify the Negative Angle Identities, you may want to have students use their calculators. For example, show that $\sin\left(\frac{\pi}{3}\right) = -\sin\left(-\frac{\pi}{3}\right)$.



Example Notes

In **Example 4,** remind the students of the earlier discussion in Section 6.4 concerning the computation of the sine and cosine of coterminal angles (page 451).

In **Example 6,** the exact value of $\sin\left(\frac{5\pi}{6}\right)$ could have just as easily been found using reference angles (pages 450–451). The solution presented in this example was made to illustrate how to use the identity $\sin t = \sin(\pi - t)$.

Negative Angle Identities

$$\sin(-t) = -\sin t$$
$$\cos(-t) = \cos t$$
$$\tan(-t) = -\tan t$$

Example 5 Negative Angle Identities

Find the exact value of $\sin\left(-\frac{\pi}{6}\right)$ and of $\cos\left(-\frac{\pi}{6}\right)$.

Solution

By the negative angle identities,

$$\sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2} \quad \text{and} \quad \cos\left(-\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

Other Identities

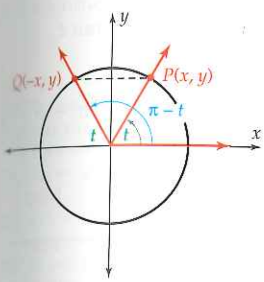


Figure 6.5-7

Let t be any real number. Figure 6.5-7 shows the angles of t and $\pi - t$ radians in standard position. The terminal side of the angle of t radians meets the unit circle at P, and the terminal side of the angle of $\pi - t$ radians meets the unit circle at Q. Congruent triangles can be used to prove what the figure illustrates:

The y-coordinates of P and Q are the same, and their x-coordinates are opposites.

This leads to the following identities.

Identities Involving $\pi - t$

$$\sin t = \sin(\pi - t)$$
$$\cos t = -\cos(\pi - t)$$
$$\tan t = -\tan(\pi - t)$$

Example 6 Identities Involving $\pi - t$

Find the exact value of $\sin\left(\frac{5\pi}{6}\right)$.

Solution

By the identity $\sin(\pi - t) = \sin t$,

$$\sin\left(\frac{5\pi}{6}\right) = \sin\left(\frac{6\pi}{6} - \frac{\pi}{6}\right) = \sin\left(\pi - \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

NOTE The identity $\sin t = \sin(\pi - t)$ is used in solving basic trigonometric equations. (See Section 8.3.)

ADDITIONAL EXAMPLES

Example 4

Find the exact value of $\cos\left(\frac{7\pi}{3}\right)$.

$$\cos\left(\frac{7\pi}{3}\right) = \cos\left(\frac{\pi}{3} + \frac{6\pi}{3}\right) = \cos\left(\frac{\pi}{3} + 2\pi\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

Example 5

Find the exact value of $\sin\left(-\frac{\pi}{3}\right)$ and of $\cos\left(-\frac{\pi}{3}\right)$.

$$\sin\left(-\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

and

$$\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

Example 6

Find the exact value of $\cos\left(\frac{2\pi}{3}\right)$.

$$\cos\left(\frac{2\pi}{3}\right) = \cos\left(\frac{3\pi}{3} - \frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

Math Background

In Chapter 7, students will be introduced to graphs of trigonometric functions. In Section 7.4 (pages 506–507), students will use graphing techniques to determine if an equation involving trigonometric functions could possibly be an identity.

Teaching Notes

The **Summary of Identities** gives a handy reference of some basic trigonometric identities. You may want to make a bulletin board display of these identities. You may also want to have students write these identities on an index card for reference. These identities are used often in the study of trigonometry and calculus. These identities may also be found on the inside back cover of this book.

Real-World Application

There is a park in Washington, D.C. called the Ellipse. If the center of the Ellipse is the origin of a coordinate system, then its boundaries can be given by the equation $\frac{x^2}{562{,}500} + \frac{y^2}{409{,}600} = 1$. By using the Pythagorean Identity $\sin^2 t + \cos^2 t = 1$ and a technique of calculus called integration, you can find the area of the park (approximately 1.5 million square feet—see Section 11.1, Exercise 23).

Exercises 6.5

ANSWERS

9. $\cos t \approx 0.9457$, $\tan t \approx 0.3438$, $\cot t \approx 2.9089$, $\sec t \approx 1.0674$, $\csc t \approx 3.0760$
10. $\sin t \approx 0.9090$, $\tan t \approx 2.1815$, $\cot t \approx 0.4584$, $\sec t \approx 2.3998$, $\csc t \approx 1.1001$
11. $\sec t \approx 3.7646$, $\cos t \approx 0.2656$, $\sin t \approx 0.9641$, $\cot t \approx 0.2755$, $\csc t \approx 1.0372$
12. $\cos t \approx 0.3869$, $\sin t \approx 0.9221$, $\tan t \approx 2.3833$, $\cot t \approx 0.4196$, $\csc t \approx 1.0845$,
13. $\sin t \approx 0.1601$, $\cos t \approx 0.9871$, $\tan t \approx 0.1622$, $\cot t \approx 6.1668$, $\sec t \approx 1.0131$
14. $\tan t \approx 0.5412$, $\sec t \approx 1.1370$, $\cos t \approx 0.8795$, $\sin t \approx 0.4759$, $\csc t \approx 2.1011$

Summary of Identities

Quotient Identities:

$$\tan t = \frac{\sin t}{\cos t} \qquad \cot t = \frac{\cos t}{\sin t}$$

Reciprocal Identities:

$$\sin t = \frac{1}{\csc t} \qquad \cos t = \frac{1}{\sec t} \qquad \tan t = \frac{1}{\cot t}$$

$$\csc t = \frac{1}{\sin t} \qquad \sec t = \frac{1}{\cos t} \qquad \cot t = \frac{1}{\tan t}$$

Pythagorean Identities:

$$\sin^2 t + \cos^2 t = 1 \qquad \tan^2 t + 1 = \sec^2 t \qquad 1 + \cot^2 t = \csc^2 t$$

Periodicity Identities:

$$\sin(t \pm 2\pi) = \sin t \qquad \cos(t \pm 2\pi) = \cos t \qquad \tan(t \pm \pi) = \tan t$$

Negative Angle Identities:

$$\sin(-t) = -\sin t \qquad \cos(-t) = \cos t \qquad \tan(-t) = -\tan t$$

Identities Involving $\pi - t$:

$$\sin t = \sin(\pi - t) \qquad \cos t = -\cos(\pi - t) \qquad \tan t = -\tan(\pi - t)$$

Exercises 6.5

In Exercises 1–4, use the quotient and reciprocal identities to simplify the given expression.

1. $\cot t \sin t$ **$\cos t$**
2. $\tan t \cot t$ **1**
3. $\csc t \sin t$ **1**
4. $\cot t \sec t$ **$\csc t$**

In Exercises 5–8, use the Pythagorean identities to simplify the given expression.

5. $\sin^2 t + \cot^2 t \sin^2 t$ **1**
6. $1 - \sec^2 t$ **$-\tan^2 t$**
7. $\dfrac{\csc^2 t - \cot^2 t}{\sin^2 t}$ **$\csc^2 t$**
8. $\dfrac{\sin^2 t - \cos^2 t \sin^2 t}{\sin^2 t}$ **$\sin^2 t$**

In Exercises 9–14, the value of one trigonometric function is given for $0 < t < \frac{\pi}{2}$. Use quotient, reciprocal, and Pythagorean identities to find the values of the remaining five trigonometric functions. Round your answers to four decimal places.

9. $\sin t = 0.3251$
10. $\cos t = 0.4167$
11. $\tan t = 3.6294$
12. $\sec t = 2.5846$
13. $\csc t = 6.2474$
14. $\cot t = 1.8479$

In Exercises 15–25, use basic identities and algebra to simplify the expression. Assume all denominators are nonzero.

15. $(\sin t + \cos t)(\sin t - \cos t)$ **$\sin^2 t - \cos^2 t$**
16. $(\sin t - \cos t)^2$ **$1 - 2\sin t\cos t$**
17. $\dfrac{\sin t}{\tan t}$ **$\cos t$**
18. $(\tan t + 2)(\tan t - 3) - (6 - \tan t) + 2\tan t$ **$\tan^2 t + 2\tan t - 12$**
19. $\left(\dfrac{4\cos^2 t}{\sin^2 t}\right)\left(\dfrac{\sin t}{4\cos t}\right)^2$ **$\frac{1}{4}$**