5.6 Solving Exponential and Logarithmic Equations

Objectives

- Solve exponential and logarithmic equations
- Solve a variety of application problems by using exponential and logarithmic equations

Exponential and logarithmic equations have been solved in this chapter so far by using the graphing method or by writing equivalent statements that can be easily solved. Most of them could also have been solved algebraically by using the techniques presented in this section, which depend primarily on the properties and laws of logarithms.

By definition of a function, if $u = v$ and f is a function, then $f(u) = f(v)$. This results in two statements.

$$\textbf{\textit{If } } u = v\textbf{, then } b^u = b^v \textbf{ for all real numbers } b > 0.$$
$$\textbf{\textit{If } } u = v\textbf{, then } \log_b u = \log_b v \textbf{ for all real numbers } b > 0.$$

Because exponential and logarithmic functions are one-to-one functions, the converse is also true.

$$\textbf{If } b^u = b^v\textbf{, then } u = v.$$
$$\textbf{If } \log_b u = \log_b v\textbf{, then } u = v.$$

Exponential Equations

The easiest exponential equations to solve are those in which both sides are powers of the same base.

Example 1 Powers of the Same Base

Solve the equation $8^x = 2^{x+1}$. Confirm your solution with a graph.

Solution

Write the equation so that each side is a power of the same base.

$$8^x = 2^{x+1}$$
$$(2^3)^x = 2^{x+1}$$
$$2^{3x} = 2^{x+1}$$
$$3x = x + 1 \qquad \textit{If } b^u = b^v\textit{, then } u = v.$$
$$2x = 1$$
$$x = \frac{1}{2}$$

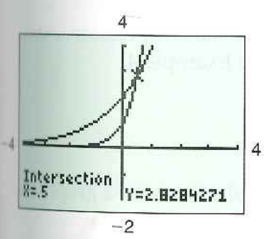


Figure 5.6-1

To find a window for the graphs of $Y_1 = 8^x$ and $Y_2 = 2^{x+1}$, consider the basic shapes of the graphs and any transformations. Because both bases of these exponential functions are greater than 1, the graphs are increasing. Because there is no vertical shift on either function, both graphs are asymptotic to the x-axis. The intersection of the graphs of $Y_1 = 8^x$ and $Y_2 = 2^{x+1}$, shown at the left, confirms the solution.

Section 5.6 Solving Exponential and Logarithmic Equations

Teaching Notes

Recall with students several equations they have solved previously by graphing:

- $6800 = 5000\left(1 + \frac{0.07}{365}\right)^t$ (page 347)
- $0.36 = 0.5^{\frac{x}{5730}}$ (page 352)
- $\frac{\ln 2}{\ln(1 + r)} = 6$ (page 361)

Tell students that they will learn methods of solving equations like these that do not require graphing.

Example Notes

Note that the title of **Example 1** is **Powers of the Same Base,** but the bases are not the same as given. Emphasize the need to *change bases so that they are the same.*

Students should confirm their solution graphically, but can also check it algebraically:

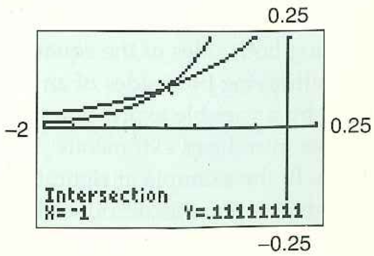
8^x	$=$ 2^{x+1}
$8^{\frac{1}{2}}$	$2^{\frac{1}{2}+1}$
$\sqrt{8}$	$2^{\frac{3}{2}}$
	$2 \cdot 2^{\frac{1}{2}}$
$2\sqrt{2}$	$2\sqrt{2}$ ✓

ADDITIONAL EXAMPLES

Example 1

Solve the equation $9^x = 3^{x-1}$. Confirm your solution with a graph.

$$9^x = 3^{x-1}$$
$$3^{2x} = 3^{x-1}$$
$$2x = x - 1$$
$$x = -1$$



ample Notes

Example 2, remind students t they can check their answer: $^{307} \approx 2.00008$

Example 4, students must ognize that $e^{2x} = (e^x)^2$ so that y can identify the equation as a adratic.

Example 5, make sure students derstand that if 58% is *lost*, % *remains*.

COMMON ERROR ALERT

udents might evaluate $\frac{\ln 3 + \ln 2}{4 \ln 2 + \ln 3}$ orrectly in **Example 3**.

- They might enter parentheses into their calculator incorrectly. Suggest that they evaluate the numerator and denominator separately before dividing.
- Other students might simplify the numerator, $\ln 3 + \ln 2$, as $\ln 5$. Remind them that $\ln 3 + \ln 2 = \ln 6$.

Example 4, some students ay incorrectly write $e^x e^x = e^{x^2}$. emind them of the **Laws of** **xponents** (page 330): $e^x = e^{x+x} = e^{2x}$, or $e^x e^x = (e^x)^2$

eaching Notes

Example 5, to help students nderstand why the function $(t) = P(0.5)^{\frac{t}{5730}}$ gives the amount carbon-14 remaining at time substitute 5730 for t: $(5730) = P(0.5)^{\frac{5730}{5730}} = P(0.5)$. This nction value confirms that if $= 5730$ years, then half of the riginal amount, or $0.5P$, remains.

Math Background

n **Example 4**, an extraneous solution introduced as a result of ultiplying both sides of the equation y e^x. Multiplying both sides of an quation by a variable expression ometimes introduces extraneous olutions. In the example at right, the xtraneous solution 0 is introduced:

$x = 3$	The only solution is 3.
$x \cdot x = 3 \cdot x$	Multiply by x.
$x^2 = 3x$	
$x^2 - 3x = 0$	The solutions are 3 and 0.

CAUTION

$\frac{\ln 2}{\ln 5} \neq \ln\left(\frac{2}{5}\right)$ and

$\frac{\ln 2}{\ln 5} \neq \ln 2 - \ln 5$

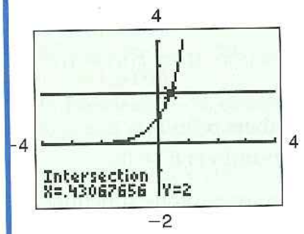


Figure 5.6-2

Example 2 Powers of Different Bases

Solve the equation $5^x = 2$. Confirm your solution with a graph.

Solution

$$5^x = 2$$
$$\ln 5^x = \ln 2 \qquad \textit{take logarithms on each side}$$
$$x \ln 5 = \ln 2 \qquad \textit{use the Power Law}$$
$$x = \frac{\ln 2}{\ln 5}$$
$$x = \frac{0.6931}{1.6094} = 0.4307$$

The intersection of the graphs of $Y_1 = 5^x$ and $Y_2 = 2$, shown at the left, confirms the solution.

Example 3 Powers of Different Bases

Solve the equation $2^{4x-1} = 3^{1-x}$. Confirm your solution with a graph.

Solution

$$2^{4x-1} = 3^{1-x}$$
$$\ln(2^{4x-1}) = \ln(3^{1-x}) \qquad \textit{Take logarithms on each side}$$
$$(4x-1)\ln 2 = (1-x)\ln 3 \qquad \textit{Power Law}$$
$$4x \ln 2 - \ln 2 = \ln 3 - x \ln 3 \qquad \textit{Distributive Property}$$
$$4x \ln 2 + x \ln 3 = \ln 3 + \ln 2 \qquad \textit{Rearrange terms and isolate } x$$
$$x(4 \ln 2 + \ln 3) = \ln 3 + \ln 2$$
$$x = \frac{\ln 3 + \ln 2}{4 \ln 2 + \ln 3}$$
$$x = 0.4628$$

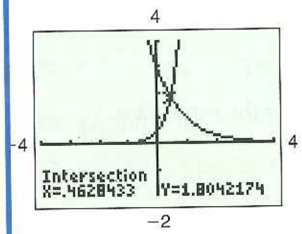


Figure 5.6-3

The intersection of the graphs of $Y_1 = 2^{4x-1}$ and $Y_2 = 3^{1-x}$, shown at the left, confirms the solution.

When you multiply each side of an equation by the same expression, extraneous solutions may be introduced, as shown in Example 4.

Example 4 Using Substitution

Solve the equation $e^x - e^{-x} = 4$. Confirm your solution with a graph.

Solution

First multiply each side by e^x to eliminate negative exponents.

$$e^x - e^{-x} = 4$$
$$e^x(e^x - e^{-x}) = e^x(4)$$
$$e^x e^x - e^x e^{-x} = 4e^x$$
$$e^{2x} - 1 = 4e^x \qquad \textit{Product Law.}$$
$$e^{2x} - 4e^x - 1 = 0$$

Let $u = e^x$ and substitute.

$$u^2 - 4u - 1 = 0$$
$$u = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)}$$
$$u = \frac{4 \pm \sqrt{20}}{2}$$
$$u = \frac{4 \pm 2\sqrt{5}}{2}$$
$$u = 2 \pm \sqrt{5}$$

Replace u with e^x to get $e^x = 2 + \sqrt{5}$ or $e^x = 2 - \sqrt{5}$. Because e^x can only be positive and $2 - \sqrt{5}$ is negative, $e^x = 2 - \sqrt{5}$ has no solution.

$$e^x = 2 + \sqrt{5}$$
$$\ln e^x = \ln(2 + \sqrt{5})$$
$$x \cdot \ln e = \ln(2 + \sqrt{5})$$
$$x = 1.4436 \qquad \ln e = 1$$

The intersection of the graphs of $Y_1 = e^x - e^{-x}$ and $Y_2 = 4$, shown in Figure 5.6-4, confirms that there is exactly one solution

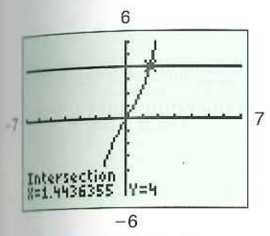


Figure 5.6-4

Applications of Exponential Equations

When a living organism dies, its carbon-14 decays. The half-life of carbon-14 is 5730 years, so the amount of carbon-14 remaining at time t is given by $M(t) = P(0.5)^{\frac{t}{5730}}$, where P is the mass of carbon-14 that was present initially. The function M can be used to determine the age of fossils and some relics.

Example 5 Radiocarbon Dating

The skeleton of a mastodon has lost 58% of its original carbon-14. When did the mastodon die?

Solution

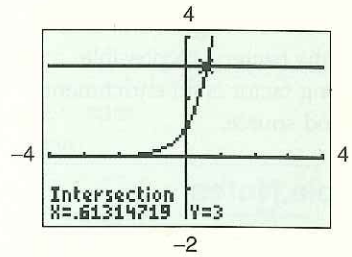
If the mastodon has lost 58% of its original carbon-14, then 42% of the initial amount, or $0.42P$, remains and $M(t) = 0.42P$. To determine when the mastodon died, solve $0.42P = P(0.5)^{\frac{t}{5730}}$ for t.

ADDITIONAL EXAMPLES

Example 2

Solve the equation $6^x = 3$. Confirm your solution with a graph.

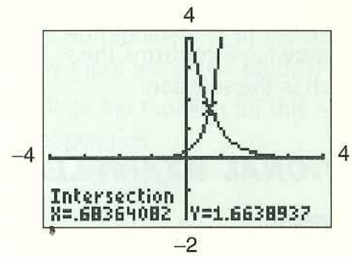
$$6^x = 3$$
$$x \ln 6 = \ln 3$$
$$x = \frac{\ln 3}{\ln 6} = 0.6131$$



Example 3

Solve the equation $4^{2x-1} = 5^{1-x}$. Confirm your solution with a graph.

$$4^{2x-1} = 5^{1-x}$$
$$(2x - 1)\ln 4 = (1 - x)\ln 5$$
$$2x \ln 4 - \ln 4 = \ln 5 - x \ln 5$$
$$x(2\ln 4 + \ln 5) = \ln 4 + \ln 5$$
$$x = \frac{\ln 4 + \ln 5}{2\ln 4 + \ln 5}$$
$$= 0.6836$$



Example 4

Solve the equation $e^x - 5e^{-x} = -6$. Confirm your solution with a graph.

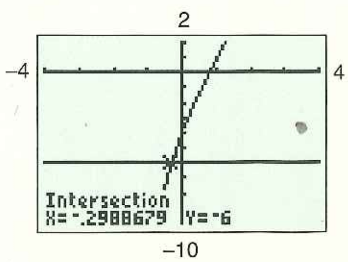
$$e^x - 5e^{-x} = -6$$
$$e^x(e^x - 5e^{-x}) = e^x(-6)$$
$$e^{2x} - 5 = -6e^x$$
$$(e^x)^2 + 6e^x - 5 = 0$$

Let $u = e^x$. $\quad u^2 + 6u - 5 = 0$

$$u = \frac{-6 \pm \sqrt{(6)^2 - 4(1)(-5)}}{2(1)}$$
$$u = \frac{-6 \pm 2\sqrt{14}}{2}$$
$$u = -3 \pm \sqrt{14}$$

e^x must be positive, so $e^x = -3 + \sqrt{14}$.

$$x \ln e = \ln(-3 + \sqrt{14})$$
$$x = -0.2989$$



Real-World Application

discussion of "inhibiting or
mulating factors" regarding
pulation growth in **Example 7** can
the foundation for the logistic
odel in **Example 8.** Some possible
hibiting factors in Example 7 are
imit on the food source or an
roduction of an agent that kills
me of the bacteria. A possible
mulating factor is an enrichment
the food source.

xample Notes

r **Example 6,** note that $\frac{10{,}680}{3000} = 3.56$
exact, not rounded. Tell students
at when they encounter a fraction
at it is *not* exact, they should
ntinue to use the fraction as they
lculate, rounding only the final
sult. Rounding later increases
curacy.

ote that the function in **Example 7,**
$t) = Pe^{rt}$, is a function of t, but also
ntains the variable r. So, to write
e required function in **part a,**
udents must first find r. Point out
at Figure 5.6-7 confirms the value of
and Figure 5.6-8 confirms the value
t, which is the solution.

DDITIONAL EXAMPLES

xample 5

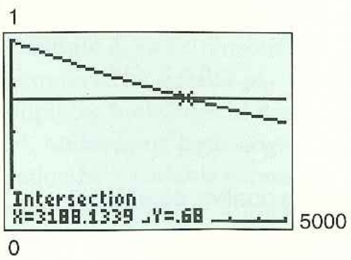
is determined that a mummy has
st 32% of its original carbon-14.
hen did the person die?

$0.68 = (0.5)^{\frac{t}{5730}}$

$0.68 = \frac{t}{5730}(\ln 0.5)$

$t = \frac{5730(\ln 0.68)}{\ln 0.5} = 3188.1339$

he person died approximately 3200
ears ago.



$$0.42P = P(0.5)^{\frac{t}{5730}}$$
$$0.42 = (0.5)^{\frac{t}{5730}}$$
$$\ln 0.42 = \ln(0.5)^{\frac{t}{5730}}$$
$$\ln 0.42 = \frac{t}{5730}(\ln 0.5)$$
$$t = \frac{5730(\ln 0.42)}{\ln 0.5}$$
$$t = 7171.3171$$

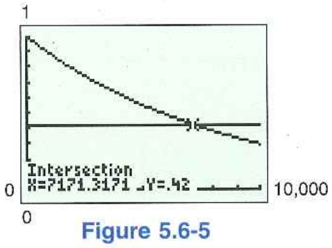


Figure 5.6-5

Therefore, the mastodon died approximately 7200 years ago. The intersection of the graphs of $Y_1 = 0.42$ and $Y_2 = (0.5)^{\frac{x}{5730}}$, shown in Figure 5.6-5, confirms the solution.

Example 6 Compound Interest

If \$3000 is to be invested at 8% per year, compounded quarterly, in how many years will the investment be worth \$10,680?

Solution

The interest rate per quarter r is $\frac{0.08}{4}$, or 0.02. To find the time t that it will take the investment to be worth \$10,680 use the compound interest formula $A = P(1 + r)^t$.

$$10{,}680 = 3000(1 + 0.02)^t$$
$$10{,}680 = 3000(1.02)^t$$
$$3.56 = (1.02)^t$$
$$\ln 3.56 = \ln(1.02)^t$$
$$\ln 3.56 = t(\ln 1.02)$$
$$t = \frac{\ln 3.56}{\ln 1.02} = 64.1208 \text{ quarters}$$

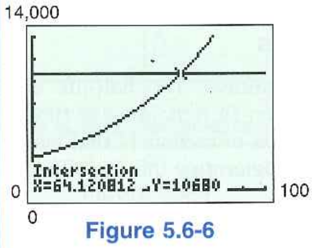


Figure 5.6-6

Therefore, it will take 64.12 quarters, or $\frac{64.12}{4} = 16.03$ years. The intersection of the graphs of $Y_1 = 10{,}680$ and $Y_2 = 3000(1 + 0.02)^x$, shown in Figure 5.6-6, confirms the solution.

Example 7 Population Growth

A biologist knows that if there are no inhibiting or stimulating factors, the population of a certain type of bacteria will increase exponentially. The population at time t is given by the function

$$S(t) = Pe^{rt},$$

where P is the initial population and r is the continuous growth rate. The biologist has a culture that contains 1000 bacteria, and 7 hours later there are 5000 bacteria.

a. Write the function for this population.

b. When will the population reach 1 billion?

Solution

a. The initial population P is 1000. To find the growth rate r, use the fact that $S(7) = 5000$.

$$S(t) = 1000e^{rt}$$
$$5000 = 1000e^{r(7)}$$
$$5 = e^{7r}$$
$$\ln 5 = \ln e^{7r}$$
$$\ln 5 = 7r \ln e$$
$$\ln 5 = 7r \qquad \ln e = 1$$
$$r = \frac{\ln 5}{7} = 0.2299$$

Therefore, the function for this population is

$$S(t) = 1000e^{0.2299t}$$

The intersection of the graphs of $Y_1 = 5000$ and $Y_2 = 1000e^{7x}$, shown in Figure 5.6-7, confirms the value of r.

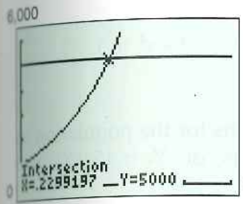


Figure 5.6-7

b. Find the value of t when $S(t)$ is 1 billion.

$$1000e^{0.2299t} = 1{,}000{,}000{,}000$$
$$e^{0.2299t} = 1{,}000{,}000$$
$$\ln e^{0.2299t} = \ln 1{,}000{,}000$$
$$0.2299t \ln e = \ln 1{,}000{,}000$$
$$0.2299t = \ln 1{,}000{,}000 \qquad \ln e = 1$$
$$t = \frac{\ln 1{,}000{,}000}{0.2299} = 60.0936 \text{ hours}$$

The bacteria population will reach 1 billion after about 60 hours. The intersection of the graphs of $Y_1 = 1000e^{0.2299x}$ and $Y_2 = 1{,}000{,}000{,}000$, shown in Figure 5.6-8, confirms the solution.

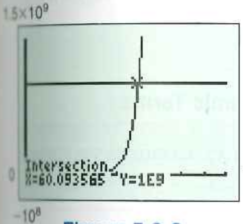


Figure 5.6-8

Example 8 Inhibited Population Growth

A population of fish in a lake at time t months is given by the function F.

$$F(t) = \frac{20{,}000}{1 + 24e^{-\frac{t}{4}}}$$

How long will it take for the fish population to reach 15,000?

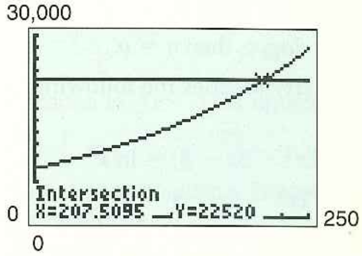
ADDITIONAL EXAMPLES

Example 6

If \$8000 is to be invested at 6% per year, compounded monthly, in how many years will the investment be worth \$22,520?

$$22{,}520 = 8000(1 + 0.005)^t$$
$$2.815 = 1.005^t$$
$$\ln 2.815 = t \ln 1.005$$
$$t = \frac{\ln 2.815}{\ln 1.005} = 207.5095 \text{ months}$$

17.29 years



Example 7

The population of a certain species of rabbit increases exponentially according to the function $S(t) = Pe^{rt}$, where P is the initial population and r is the continuous growth rate. At the beginning of a 12-month study, there were 50 rabbits, and at the end of the study there were 164 rabbits.

a. Write the function for this population.

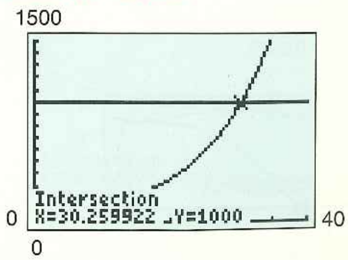
$$164 = 50e^{r(12)}$$
$$3.28 = e^{12r}$$
$$\ln 3.28 = 12r \ln e$$
$$r = \frac{\ln 3.28}{12} = 0.990$$

So, $S(t) = 50e^{0.099t}$.

b. When will the population reach 1000?

$$1000 = 50e^{0.099t}$$
$$20 = e^{0.099t}$$
$$\ln 20 = 0.099t \ln e$$
$$t = \frac{\ln 20}{0.099} = 30.2599$$

about 30 months



Solution

Find the value of t when $F(t)$ is 15,000.

$$15{,}000 = \frac{20{,}000}{1 + 24e^{-\frac{t}{4}}}$$
$$15{,}000(1 + 24e^{-\frac{t}{4}}) = 20{,}000$$
$$1 + 24e^{-\frac{t}{4}} = \frac{20{,}000}{15{,}000}$$
$$24e^{-\frac{t}{4}} = \frac{4}{3} - 1$$
$$e^{-\frac{t}{4}} = \frac{1}{3} \cdot \frac{1}{24}$$
$$\ln e^{-\frac{t}{4}} = \ln \frac{1}{72}$$
$$-\frac{t}{4} \ln e = \ln 1 - \ln 72$$
$$-\frac{t}{4} = 0 - \ln 72 \qquad \ln e = 1 \text{ and } \ln 1 = 0$$
$$t = 4 \ln 72 = 17.1067$$

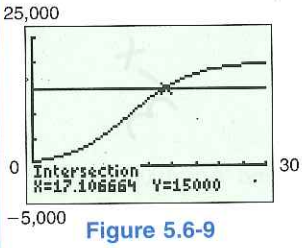


Figure 5.6-9

Therefore, it will take a little more than 17 months for the population to reach 15,000. The intersection of the graphs of $Y_1 = 15{,}000$ and $Y_2 = \frac{20{,}000}{1 + 24e^{-\frac{x}{4}}}$ shown in Figure 5.6-9 confirms the solution.

Logarithmic Equations

Properties of one-to-one functions are useful when solving logarithmic equations, as shown in Example 9.

Example 9 Equations with Only Logarithmic Terms

Solve the equation $\ln(x - 3) + \ln(2x + 1) = 2(\ln x)$. Confirm your solution with a graph.

Solution

First use the Product and Power Laws to rewrite the equation.

$$\ln(x - 3) + \ln(2x + 1) = 2(\ln x)$$
$$\ln[(x - 3)(2x + 1)] = \ln x^2$$
$$\ln(2x^2 - 5x - 3) = \ln x^2$$
$$2x^2 - 5x - 3 = x^2 \qquad y = \ln x \text{ is a one-to-one function}$$
$$x^2 - 5x - 3 = 0$$

Example Notes

Figure 5.6-9 shows that the function in Example 8 is a logistic model. Rather than increasing exponentially, the growth levels off. Students might recall from Section 5.2 (pages 341–342) that a logistic model is designed for situations in which future growth is limited due to a fixed area, food supply, or other factors.

When discussing Example 9, remind students that the following property is true because logarithmic functions are one-to-one:

If $\log_b u = \log_b v$, then $u = v$.

This property justifies the following step:

If $\ln(2x^2 - 5x - 3) = \ln x^2$

then $2x^2 - 5x - 3 = x^2$

Note that if students do not realize that $\ln(x - 3)$ is undefined for $x \le 3$, they *will* realize it by checking their solutions. By entering $\ln(-0.5414 - 3)$ on a calculator, they will get an ERROR message.

ADDITIONAL EXAMPLES

Example 8

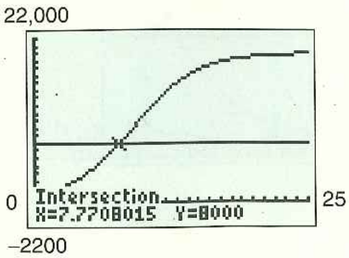
The population of a certain bacteria culture at time t hours is given by the function F.

$$F(t) = \frac{20{,}000}{1 + 20e^{-\frac{t}{3}}}$$

How long will it take for the bacteria population to reach 8000?

$$8000 = \frac{20{,}000}{1 + 20e^{-\frac{t}{3}}}$$
$$1 + 20e^{-\frac{t}{3}} = \frac{20{,}000}{8000}$$
$$20e^{-\frac{t}{3}} = 2.5 - 1$$
$$e^{-\frac{t}{3}} = \frac{1.5}{20}$$
$$-\frac{t}{3} \ln e = \ln(0.075)$$
$$t = -3 \ln(0.075) = 7.7708$$

about 7.8 hours



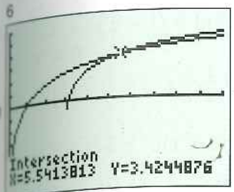
Use the Quadratic Formula to solve for x.

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-3)}}{2(1)}$$

$$x = \frac{5 + \sqrt{37}}{2} \approx 5.5414 \quad \text{or} \quad x = \frac{5 - \sqrt{37}}{2} \approx -0.5414$$

Because $\ln(x - 3)$ is undefined for $x \le 3$, $x = \frac{5 - \sqrt{37}}{2} \approx -0.5414$ cannot be a solution. Therefore, the only solution of the original equation is $x = \frac{5 + \sqrt{37}}{2} \approx 5.5414$. The intersection of the graphs of $Y_1 = \ln(x - 3) + \ln(2x + 1)$ and $Y_2 = 2(\ln x)$, shown in Figure 5.6-10, confirms the solution.

6



0 10 −6

Figure 5.6-10

Equations that involve both logarithmic and constant terms may be solved by using the basic property of logarithms.

$$10^{\log v} = v \quad \text{and} \quad e^{\ln v} = v$$

Example 10 Equations with Logarithmic and Constant Terms

Solve the equation $\ln(x - 3) = 5 - \ln(x - 3)$. Confirm your solution with a graph.

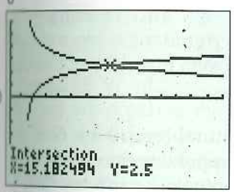
Solution

First get all the logarithmic terms on one side of the equal sign and the constants on the other. Then rewrite the side that contains the logarithms as a single logarithm.

$$\ln(x - 3) = 5 - \ln(x - 3)$$
$$\ln(x - 3) + \ln(x - 3) = 5$$
$$2\ln(x - 3) = 5$$
$$\ln(x - 3) = \frac{5}{2}$$
$$e^{\ln(x-3)} = e^{\frac{5}{2}}$$
$$x - 3 = e^{\frac{5}{2}}$$
$$x = e^{\frac{5}{2}} + 3 \approx 15.1825$$

The intersection of the graphs of $Y_1 = \ln(x - 3)$ and $Y_2 = 5 - \ln(x - 3)$, shown in Figure 5.6-11, confirms the solution.

6



0 30 −6

Figure 5.6-11

Example 11 Equations with Logarithmic and Constant Terms

Solve the equation $\log(x - 16) = 2 - \log(x - 1)$. Confirm your solution with a graph.

ADDITIONAL EXAMPLES

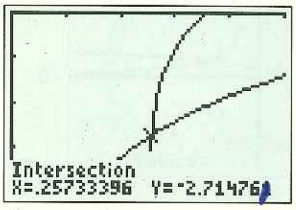
Example 9

Solve the equation
$\ln(4x - 1) + \ln(x + 2) = 2(\ln x)$.
Confirm your solution with a graph.

$$\ln(4x - 1) + \ln(x + 2) = 2(\ln x)$$
$$\ln[(4x - 1)(x + 2)] = 2(\ln x)$$
$$\ln(4x^2 + 7x - 2) = \ln x^2$$
$$4x^2 + 7x - 2 = x^2$$
$$3x^2 + 7x - 2 = 0$$
$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(3)(-2)}}{2(3)}$$
$$= \frac{-7 \pm \sqrt{73}}{6}$$

Because $\ln(4x - 1)$ is undefined for $x \le \frac{1}{4}$, $x = \frac{-7 - \sqrt{73}}{6} \approx -2.5907$ cannot be a solution. Therefore, the only solution is $x = \frac{-7 + \sqrt{73}}{6} \approx 0.2573$

0



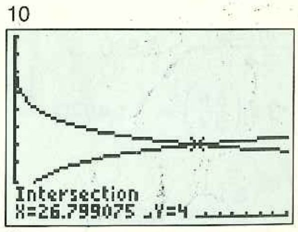
0 0.5 −4

Example 10

Solve the equation
$\ln(2x + 1) = 8 - \ln(2x + 1)$.
Confirm your solution with a graph.

$$\ln(2x + 1) = 8 - \ln(2x + 1)$$
$$2\ln(2x + 1) = 8$$
$$\ln(2x + 1) = 4$$
$$e^{\ln(2x+1)} = e^4$$
$$2x + 1 = e^4$$
$$2x = e^4 - 1$$
$$x = \frac{e^4 - 1}{2}$$
$$\approx 26.7991$$

10



0 40 0

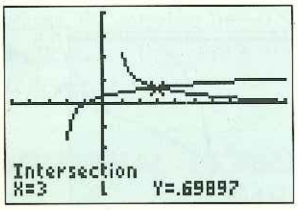
ADDITIONAL EXAMPLES

Example 11

Solve the equation
$\log(x + 2) = 1 - \log(x - 1)$.
Confirm your solution with a graph.

$$\log(x+2) = 1 - \log(x-1)$$
$$\log(x+2) + \log(x-1) = 1$$
$$\log[(x+2)(x-1)] = 1$$
$$\log(x^2 + x - 2) = 1$$
$$10^{\log(x^2+x-2)} = 10^1$$
$$x^2 + x - 2 = 10$$
$$x^2 + x - 12 = 0$$
$$(x+4)(x-3) = 0$$
$$x = -4;\ x = 3$$

Because $\log(x + 2)$ and $\log(x - 1)$ are not defined for $x = -4$, it cannot be a solution. Therefore, the only solution is $x = 3$.



Solution

$$\log(x - 16) = 2 - \log(x - 1)$$
$$\log(x - 16) + \log(x - 1) = 2$$
$$\log[(x - 16)(x - 1)] = 2$$
$$\log(x^2 - 17x + 16) = 2$$
$$10^{\log(x^2-17x+16)} = 10^2$$
$$x^2 - 17x + 16 = 100$$
$$x^2 - 17x - 84 = 0$$
$$(x + 4)(x - 21) = 0$$
$$x + 4 = 0 \quad \text{or} \quad x - 21 = 0$$
$$x = -4 \qquad x = 21$$

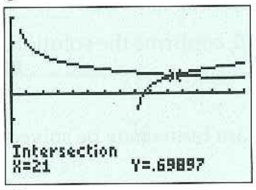


Figure 5.6-12

Because $\log(x - 16)$ and $\log(x - 1)$ are not defined for $x = -4$, it cannot be a solution. Therefore, the only solution is $x = 21$. The intersection of the graphs of $Y_1 = \log(x - 16)$ and $Y_2 = 2 - \log(x - 1)$, shown in Figure 5.6-12, confirms the solution.

1-5, 7-18, 35-42, 53-62

Exercises 5.6

In Exercises 1–8, solve the equation without using logarithms.

1. $3^x = 81$ **$x = 4$**
2. $3^x + 3 = 30$ **$x = 3$**
3. $3^{x+1} = 9^{5x}$ **$x = \frac{1}{9}$**
4. $4^{5x} = 16^{2x-1}$ **$x = -2$**
5. $3^{5x}9^{x^2} = 27$ **$x = \frac{1}{2}$ or -3**
6. $2^{x^2+5x} = \frac{1}{16}$ **$x = -1$ or -4**
7. $9^{x^2} = 3^{-5x-2}$ **$x = -2$ or $-\frac{1}{2}$**
8. $4^{x^2-1} = 8^x$ **$x = -\frac{1}{2}$ or 2**

In Exercises 9–29, solve the equation. Give exact answers (in terms of natural logarithms). Then use a calculator to find an approximate answer.

9. $3^x = 5$
10. $5^x = 4$ **$x = \frac{\ln 4}{\ln 5} \approx 0.8614$**
11. $2^x = 3^{x-1}$
12. $4^{x+2} = 2^{x-1}$ **$x = \frac{-5\ln 2}{\ln 2} = -5$**
13. $3^{1-2x} = 5^{x+5}$
14. $4^{3x-1} = 3^{x-2}$
15. $2^{1-3x} = 3^{x+1}$
16. $3^{z+3} = 2^z$
17. $e^{2x} = 5$
18. $e^{-3x} = 2$ **$x = \frac{\ln 2}{-3} \approx -0.2310$**
19. $6e^{-1.4x} = 21$
20. $3.4e^{-\frac{x}{3}} = 5.6$
21. $2.1e^{\frac{x}{2}\ln 3} = 5$
22. $7.8e^{\frac{x}{3}\ln 5} = 14$
23. $9^x - 4 \cdot 3^x + 3 = 0$ *Hint:* Note that $9^x = (3^x)^2$; let $u = 3^x$. **$x = 0$ or 1**
24. $4^x - 6 \cdot 2^x = -8$ **$x = 2$ or 1**
25. $e^{2x} - 5e^x + 6 = 0$ *Hint:* Let $u = e^x$.
26. $2e^{2x} - 9e^x + 4 = 0$
27. $6e^{2x} - 16e^x = 6$
28. $8e^{2x} + 8e^x = 6$
29. $4^x + 6 \cdot 4^{-x} = 5$

In Exercises 30–32, solve the equation for x.

30. $\frac{e^x + e^{-x}}{e^x - e^{-x}} = t$
31. $\frac{e^x - e^{-x}}{2} = t$
32. $\frac{e^x - e^{-x}}{e^x + e^{-x}} = t$

33. Prove that if $\ln u = \ln v$, then $u = v$. *Hint:* Use the basic property of inverses $e^{\ln v} = v$.
34. a. Solve $7^x = 3$ using natural logarithms. Give an exact answer, not an approximation.
    b. Solve $7^x = 3$ using common logarithms. Give an exact answer, not an approximation.
    c. Use the change-of-base formula in *Excursion* 5.5.A to show that your answers in parts **a** and **b** are the same.

In Exercises 35–44, solve the equation. (See Example 9.)

35. $\ln(3x - 5) = \ln 11 + \ln 2$ **$x = 9$**
36. $\log(4x - 1) = \log(x + 1) + \log 2$ **$x = \frac{3}{2}$**

Exercises 5.6

ANSWERS

9. $x = \frac{\ln 5}{\ln 3} \approx 1.465$
11. $x = \frac{\ln 3}{\ln 1.5} \approx 2.7095$
13. $x = \frac{\ln 3 - 5\ln 5}{\ln 5 + 2\ln 3} \approx -1.825$
14. $x = \frac{\ln 4 - 2\ln 3}{3\ln 4 - \ln 3} \approx -0.2650$
15. $x = \frac{\ln 2 - \ln 3}{3\ln 2 + \ln 3} \approx -0.1276$
16. $z = \frac{-3\ln 3}{\ln 3 - \ln 2} \approx -8.1285$
17. $x = \frac{(\ln 5)}{2} \approx 0.805$
19. $x = \frac{(-\ln 3.5)}{1.4} \approx -0.895$
20. $x = -3\ln\left(\frac{5.6}{3.4}\right) \approx -1.4970$
21. $x = \frac{2\ln\left(\frac{5}{2.1}\right)}{\ln 3} \approx 1.579$
22. $x = \frac{3\ln\left(\frac{14}{7.8}\right)}{\ln 5} \approx 1.0903$
25. $x = \ln 2 \approx 0.693$ or $x = \ln 3 \approx 1.099$
26. $x = \ln\left(\frac{1}{2}\right) \approx -0.693$ or $x = \ln 4 \approx 1.386$
27. $x = \ln 3 \approx 1.099$
28. $x = \ln\left(\frac{1}{2}\right) \approx -0.693$
29. $x = \frac{\ln 2}{\ln 4} = \frac{1}{2}$ or $x = \frac{\ln 3}{\ln 4} \approx 0.792$
30. $x = \ln\sqrt{\frac{1+t}{t-1}}$
31. $x = \ln(t + \sqrt{t^2 + 1})$
32. $x = \ln\sqrt{\frac{t+1}{1-t}}$
33. If $\ln u = \ln v$, then $e^{\ln u} = e^{\ln v}$, so $u = v$.
34. a. $x = \frac{\ln 3}{\ln 7}$
    b. $x = \frac{\log 3}{\log 7}$
    c. $x = \frac{\ln 3}{\ln 7}$