51. a. The half-life of polonium-210 is 140 days. Find the rule of the function that gives the amount of polonium-210 remaining from an initial 20 milligrams after t days. $f(t) \approx 20(0.5^{\frac{t}{140}})$

b. How much polonium-210 is left after 15 weeks? after 52 weeks?
≈11.892 mg; ≈3.299 mg

c. How long will it take for the 20 milligrams to decay to 4 milligrams?
≈325 days

52. How old is a piece of ivory that has lost 58% of its carbon-14? (See Example 9.)
≈7171 years old

53. How old is a mummy that has lost 49% of its carbon-14?
≈5566 years old

5.4 Common and Natural Logarithmic Functions

Objectives

- Evaluate common and natural logarithms with and without a calculator
- Solve common and natural exponential and logarithmic equations by using an equivalent equation
- Graph and identify transformations of common and natural logarithmic functions

Technology Tip

The graph of $f(x) = 10^x$ can be obtained in parametric mode by letting

$$x = t \quad \text{and} \quad y = 10^t,$$

where t is any real number.

The graph of the inverse function g can then be obtained by letting

$$x = 10^t \quad \text{and} \quad y = t,$$

where t is any real number.

From their invention in the seventeenth century until the development of computers and calculators, logarithms were the only effective tools for numerical computation in astronomy, chemistry, physics, and engineering. Although they are no longer needed for computation, logarithmic functions still play an important role in the sciences and engineering. In this section you will examine the two most important types of logarithms, those to base 10 and those to base e. Logarithms to other bases are considered in *Excursion* 5.5A.

Common Logarithms

The graph of the exponential function $f(x) = 10^x$ is shown in Figure 5.4-1. Because it is an increasing function, it is a one-to-one function, as explained in Section 3.6. Recall that the graphs of inverse functions are reflections of one another across the line $y = x$. The exponential function $f(x) = 10^x$ and its inverse function are graphed in Figure 5.4-2.

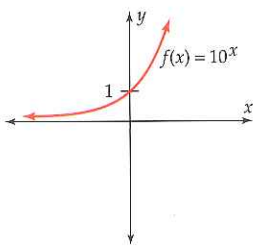


Figure 5.4-1

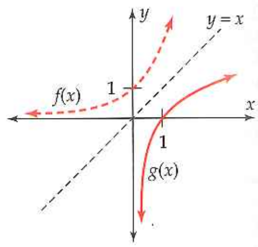


Figure 5.4-2

The inverse function of the exponential function $f(x) = 10^x$ is called the **common logarithmic function.** The value of this function at the number x is denoted as $\log x$ and called the common logarithm of the number x.

The functions $f(x) = 10^x$ and $g(x) = \log x$ are inverse functions.

$$\log v = u \quad \textbf{if and only if} \quad 10^u = v$$

Section 5.4 Common and Natural Logarithmic Functions

Teaching Notes

Remind students that if a function is one-to-one, its inverse is also a function.

To further illustrate the relationship between $f(x) = 10^x$ and $g(x) = \log x$, students can create a table of values for the function $f(x) = 10^x$ as shown.

x	-1	0	1	2
$f(x) = 10^x$	$\frac{1}{10}$	1	10	100

Students may then create another table of values for the inverse by switching values and renaming the function as shown.

x	$\frac{1}{10}$	1	10	100
$g(x) = \log x$	-1	0	1	2

Have students copy and complete the table below to reinforce the definition of the **common logarithmic function:**

$\log v = u$	is equivalent to	$10^u = v$
$\log 1000 = 3$		$10^3 = 1000$
$\log 100 = 2$		$10^2 = 100$
$\log 10 = 1$		$10^1 = 10$
$\log 1 = 0$		$10^0 = 1$
$\log \frac{1}{10} = -1$		$10^{-1} = \frac{1}{10}$
$\log \frac{1}{100} = -2$		$10^{-2} = \frac{1}{100}$
$\log \frac{1}{1000} = -3$		$10^{-3} = \frac{1}{1000}$

Because logarithms are a special kind of exponent, every statement about logarithms is equivalent to a statement about exponents.

Logarithmic statement $\log v = u$	Equivalent exponential statement $10^u = v$
$\log 29 = 1.4624$	$10^{1.4624} = 29$
$\log 378 = 2.5775$	$10^{2.5775} = 378$

Example 1 Evaluating Common Logarithms

Without using a calculator, find each value.

a. $\log 1000$ **b.** $\log 1$ **c.** $\log \sqrt{10}$ **d.** $\log(-3)$

Solution

a. If $\log 1000 = x$, then $10^x = 1000$. Because $10^3 = 1000$, $\log 1000 = 3$.

b. If $\log 1 = x$, then $10^x = 1$. Because $10^0 = 1$, $\log 1 = 0$.

c. If $\log \sqrt{10} = x$, then $10^x = \sqrt{10}$. Because $10^{\frac{1}{2}} = \sqrt{10}$, $\log \sqrt{10} = \frac{1}{2}$.

d. If $\log(-3) = x$, then $10^x = -3$. Because there is no real number exponent of 10 that produces -3, $\log(-3)$ is not defined for real numbers.

…TE Logarithms are …ded to four decimal …s and an equal sign is … rather than the …roximately equal" … The word "common" …be omitted except … it is necessary to …nguish the common …ithm from another …of logarithm.

Every scientific and graphing calculator has a LOG key for evaluating logarithms. For example,

$$\log 0.6 = -0.2218 \quad \text{and} \quad \log 327 = 2.5145$$

A calculator is necessary to evaluate most logarithms, but you can get a rough estimate mentally. For example, because log 795 is greater than log 100 = 2 and less than log 1000 = 3, you can estimate that log 795 is between 2 and 3 and closer to 3.

Example 2 Using Equivalent Statements

Solve each equation by using an equivalent statement.

a. $\log x = 2$ **b.** $10^x = 29$

$\log_{10} 29$

Solution

a. If $\log x = 2$, then $10^2 = x$. Therefore, $x = 100$.

b. If $10^x = 29$, then $\log 29 = x$. Therefore, $x = 1.4624$, as shown in Figure 5.4-3.

```
(29)
 1.462397998
```

Figure 5.4-3

Example Notes

Before discussing **Example 1d**, ask students what exponent x makes $10^x = -3$ true. There is none. This should help students understand why $\log(-3)$ is undefined.

To help students find equivalent statements in **Example 2**, align each with the definition:

$\log v = u$ if and only if $10^u = v$

$\log x = 2$ if and only if $10^2 = x$

$\log 29 = x$ if and only if $10^x = 29$

ADDITIONAL EXAMPLES

Example 1

Without using a calculator, find each value.

a. $\log 100{,}000$ 5

b. $\log 10$ 1

c. $\log \sqrt[3]{10}$ $\frac{1}{3}$

d. $\log(-7)$ undefined for real numbers

Example 2

Solve each equation by using an equivalent statement.

a. $\log x = 5$

$10^5 = x$

$x = 100{,}000$

b. $10^x = 52$

$\log 52 = x$

$x = 1.7160$

```
log(52)
 1.716003344
```

$10^2 = x$

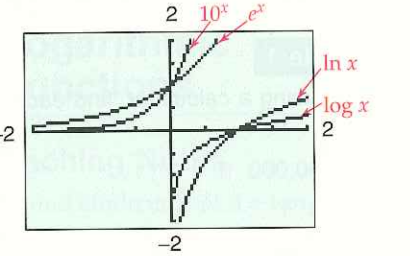
$x = 100$

1,462397948

eaching Notes

ote to students that the graphs of the verse pair $f(x) = e^x$ and $g(x) = \ln x$ n this page closely resemble the raphs of the inverse pair $f(x) = 10^x$ nd $g(x) = \log x$ on page 356. mphasize that they *are* similar in hape and properties, but they are ot identical.

lave students graph these four unctions on a graphing calculator sing a viewing window with $-2 \le x \le 2$ and $-2 \le y \le 2$ to bserve this.



Remind students of the examples hey wrote to illustrate common ogarithms:

og 1000 = 3 is equivalent to $10^3 = 1000$
og 100 = 2 is equivalent to $10^2 = 100$
etc.

Have them rewrite the logarithms as ollows:

og $10^3 = 3$
og $10^2 = 2$
etc.

Now have them write these natural ogarithms:

n $e^3 = 3$
n $e^2 = 2$
n $e^1 = 1$ ($\ln e = 1$)
n $e^0 = 0$ ($\ln 1 = 0$)
ln $e^{-1} = -1$ ($\ln \frac{1}{e} = -1$)
ln $e^{-2} = -2$ ($\ln \frac{1}{e^2} = -2$)

Students often have more trouble with natural logarithms than with common logarithms because of the bases; 10 is a whole number and e is irrational. Have them use their calculator to approximate e^3 and e^2 to the nearest whole number. 20, 7 Then have them copy and complete the following statements, using their calculators:

$3 = \ln e^3 \approx \ln 20$; $\ln 20 \approx$ 2.996
$2 = \ln e^2 \approx \ln 7$; $\ln 7 \approx$ 1.950

Natural Logarithms

The exponential function $f(x) = e^x$ is very useful in science and engineering. Consequently, another type of logarithm exists, based on the number e instead of 10.

The graph of the exponential function $f(x) = e^x$ is shown in Figure 5.4-4. Because it is an increasing function, it is one-to-one. The function $f(x) = e^x$ and its inverse function are graphed in Figure 5.4-5.

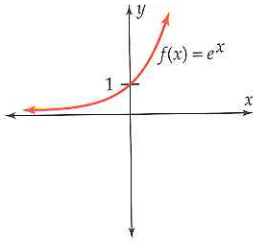


Figure 5.4-4

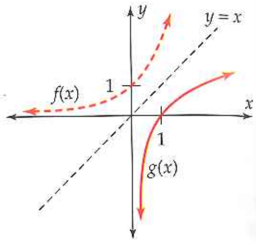


Figure 5.4-5

This inverse function of the exponential function $f(x) = e^x$ is called the **natural logarithmic function.** The value of this function at the number x is denoted as $\ln x$ and called the **natural logarithm** of the number x.

The functions $f(x) = e^x$ and $g(x) = \ln x$ are inverse functions.

$$\ln v = u \quad \textbf{if and only if} \quad e^u = v$$

Again, as with common logarithms, every statement about natural logarithms is equivalent to a statement about exponents.

Logarithmic statement $\ln v = u$	Equivalent exponential statement $e^u = v$
$\ln 14 = 2.6391$	$e^{2.6391} = 14$
$\ln 0.2 = -1.6094$	$e^{-1.6094} = 0.2$

Example 3 Evaluating Natural Logarithms

Use a calculator to find each value.

a. ln 0.15 **b.** ln 186 **c.** ln (−5)

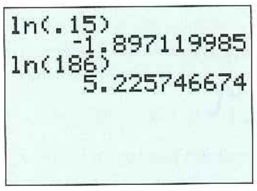


Figure 5.4-6

Solution

a. $\ln 0.15 = -1.8971$, which means that $e^{-1.8971} = 0.15$.

b. $\ln 186 = 5.2257$, which means that $e^{5.2257} \doteq 186$.

c. $\ln(-5)$ is undefined for real numbers because there is no exponent of e that produces -5.

In a few cases you can evaluate $\ln x$ without a calculator.

$$\ln e = 1 \quad \text{because} \quad e^1 = e$$
$$\ln 1 = 0 \quad \text{because} \quad e^0 = 1$$

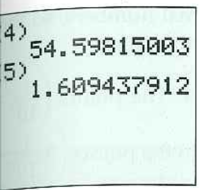


Figure 5.4-7

Example 4 Solving by Using an Equivalent Statement

Solve each equation by using an equivalent statement.

a. $\ln x = 4$ **b.** $e^x = 5$

Solution

a. If $\ln x = 4$, then $e^4 = x$. Therefore, $x = 54.5982$.
b. If $e^x = 5$, then $\ln 5 = x$. Therefore, $x = 1.6094$.

Graphs of Logarithmic Functions

Because the graphs of exponential functions have the same basic shape and each logarithmic function is the inverse of an exponential function, the graphs of logarithmic functions have common characteristics.

The following table compares the graphs of exponential and logarithmic functions.

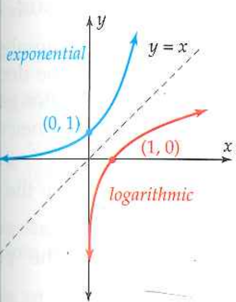


Figure 5.4-8

	Exponential functions	Logarithmic functions
Examples	$f(x) = 10^x$; $f(x) = e^x$	$g(x) = \log x$; $g(x) = \ln x$
Domain	all real numbers	all positive real numbers
Range	all positive real numbers	all real numbers
	$f(x)$ increases as x increases	$g(x)$ increases as x increases
	$f(x)$ approaches the x-axis as x decreases	$g(x)$ approaches the y-axis as x approaches 0
Reference points	$f(x) = 10^x$ $\left(-1, \frac{1}{10}\right)$, (0, 1), (1, 10)	$g(x) = \log x$ $\left(\frac{1}{10}, -1\right)$, (1, 0), (10, 1)
	$f(x) = e^x$ $\left(-1, \frac{1}{e}\right)$, (0, 1), (1, e)	$g(x) = \ln x$ $\left(\frac{1}{e}, -1\right)$, (1, 0), ($e$, 1)

Example 5 Transforming Logarithmic Functions

Describe the transformation from the graph of $g(x) = \log x$ to the graph of $h(x) = 2\log(x - 3)$. Give the domain and range of h.

ADDITIONAL EXAMPLES

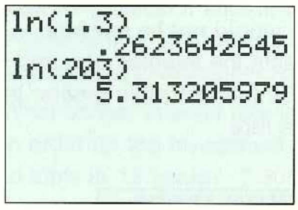
Example 3

Use a calculator to find each value.

a. ln 1.3 0.2624

b. ln 203 5.3132

c. ln (−12) undefined for real numbers



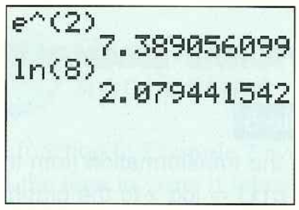
Example 4

Solve each equation by using an equivalent statement.

a. $\ln x = 2$ **b.** $e^x = 8$

$e^2 = x$ $\ln 8 = x$

$x = 7.3891$ $x = 2.0794$



Teaching Notes

For the discussion of **Graphs of Logarithmic Functions,** make sure that students understand the following:

- $10^0 = 1$ and $e^0 = 1$, so both exponential graphs 10^x and e^x pass through the point (0, 1). The blue exponential graph in Figure 5.4-8 could represent either exponential function.
- $\log 1 = 0$ and $\ln 1 = 0$, so both logarithmic graphs $\log x$ and $\ln x$ pass through the point (1, 0). The red logarithmic graph in Figure 5.4-8 could represent either logarithmic function.

Ask students:
Why is the domain of an exponential function the range of its corresponding logarithmic function? because the functions are inverses

What is the asymptote for each graph? y-axis for the logarithmic function and x-axis for the exponential function

Example Notes

or **Example 5,** review horizontal hifts (page 175) and vertical stretches page 179).

or **Example 6,** review reflections page 177) and vertical shifts page 174).

n the graph for **Example 7b,** ask udents why x-values greater than 1 robably would not be relevant. represents the interest rate, and = 100%. Interest rates greater than 00% are rare.

COMMON ERROR ALERT

Vhen entering a function such as the ne in **Example 7b** into a calculator, it important that students include arentheses around the appropriate xpression.

(2/ln(1 + x) incorrect

(2)/ln(1 + x) correct

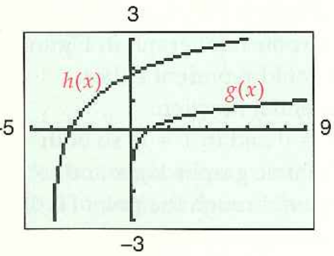
ADDITIONAL EXAMPLES

Example 5

escribe the transformation from the raph of $g(x) = \log x$ to the graph of $(x) = 3\log(x + 4)$. Give the domain nd range of h. The graph of h is he graph of g after a horizontal ranslation of 4 units left and a ertical stretch by a factor of 3.

omain of h: all real numbers greater han -4

ange of h: all real numbers



Solution

The graph of $h(x) = 2 \cdot g(x - 3)$ is the graph of $g(x) = \log x$ after a horizontal translation of 3 units right and a vertical stretch by a factor of 2.

Domain of h: The domain of $g(x) = \log x$ is all positive real numbers. The horizontal translation of 3 units to the right changes the domain to all real numbers greater than 3.

Range of h: The range of $g(x) = \log x$ is all real numbers, so the vertical stretch has no effect on the range.

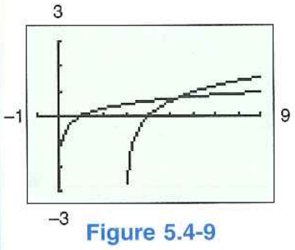


Figure 5.4-9

The graphs of g and h are shown in Figure 5.4-9. The points $\left(\frac{1}{10}, -1\right)$, $(1, 0)$, and $(10, 1)$ on the graph of g are translated to the points $\left(3\frac{1}{10}, -2\right)$, $(4, 0)$, and $(13, 2)$ on the graph of h. Although the graph of $h(x) = 2\log(x - 3)$ appears to stop abruptly at $x = 3$, you know that it continues to approach the asymptote at $x = 3$.

Example 6 Transforming Logarithmic Functions

Describe the transformation from $g(x) = \ln x$ to $h(x) = \ln(2 - x) - 3$. Give the domain and range of h.

Solution

Because $h(x) = g(-(x - 2)) - 3$, its graph is that of $g(x) = \ln x$ after a horizontal reflection across the y-axis followed by a horizontal translation of 2 units to the right and a vertical translation of 3 units downward.

Domain of h: The domain of $g(x) = \ln x$ is all positive real numbers. The reflection across the y-axis first changes the domain to all negative real numbers. Then the translation of 2 units to the right changes the domain from all negative real numbers to all real numbers less than 2.

Range of h: The range of $g(x) = \ln x$ is all real numbers, so the vertical translation does not affect the range.

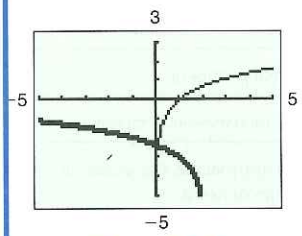


Figure 5.4-10

The graphs of g and h are shown in Figure 5.4-10. The points $\left(\frac{1}{e}, -1\right)$, $(1, 0)$, and $(e, 1)$ on the graph of g are translated to points $\left(2 - \frac{1}{e}, -4\right)$, $(1, -3)$, and $(2 - e, -2)$ on the graph of h.

Example 7 Solving Logarithmic Equations Graphically

If you invest money at an interest rate r, compounded annually, then $D(r)$ gives the time in years that it would take to double.

$$D(r) = \frac{\ln 2}{\ln(1 + r)}$$

Example 6

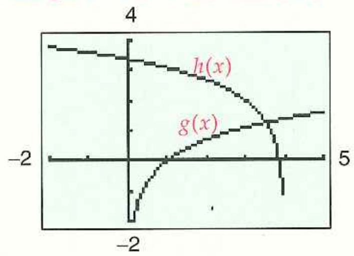
Describe the transformation from the graph of $g(x) = \ln x$ to the graph of $h(x) = \ln(4 - x) + 2$. Give the domain and range of h.

$h(x) = g(-(x - 4)) + 2$

The graph of h is the graph of g after a horizontal reflection across the y-axis followed by a horizontal translation of 4 units right and a vertical translation of 2 units upward.

Domain of h: all real numbers less than 4

Range of h: all real numbers



a. How long will it take to double an investment of $2500 at 6.5% annual interest?

b. What annual interest rate is needed in order for the investment in part a to double in 6 years?

Solution

a. The annual interest rate r is 0.065. Find $D(0.065)$.

$$D(0.065) = \frac{\ln 2}{\ln(1 + 0.065)} = 11.0067$$

Therefore, it will take approximately 11 years to double an investment of $2500 at 6.5% annual interest.

b. If the investment doubles in 6 years, then $D(r) = 6$. To find the annual interest rate r, solve $\frac{\ln 2}{\ln(1 + r)} = 6$ by graphing. The point of intersection of the graphs of $Y_1 = \frac{\ln 2}{\ln(1 + r)}$ and $Y_2 = 6$ is approximately (0.1225, 6). Therefore, an annual interest rate of 12.25% is needed for the investment to double in 6 years. See Figure 5.4-11.

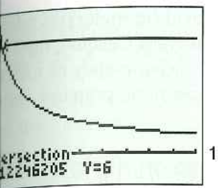


Figure 5.4-11

ercises 5.4

ess stated otherwise, all letters represent positive nbers.

xercises 1–4, find the value of each logarithm.

log 10,000 **4**

2. log 0.001 **−3**

$\log \frac{\sqrt{10}}{1000}$ **−2.5**

4. $\log \sqrt[3]{0.01}$ **$-\frac{2}{3}$**

xercises 5–14, translate the given logarithmic state- t into an equivalent exponential statement.

log 1000 = 3 **$10^3 = 1000$**

6. log 0.001 = −3 **$10^{-3} = 0.001$**

log 750 = 2.8751 **$10^{2.8751} = 750$**

8. log 0.8 = −0.0969 **$10^{-0.0969} = 0.8$**

ln 3 = 1.0986 **$e^{1.0986} = 3$**

10. ln 10 = 2.3026 **$e^{2.3026} = 10$**

ln 0.01 = −4.6052 **$e^{-4.6052} = 0.01$**

12. $\ln s = r$ **$e^r = s$**

$\ln(x^2 + 2y) = z + w$ **$e^{z+w} = x^2 + 2y$**

14. $\log(a + c) = d$ **$10^d = a + c$**

xercises 15–24, translate the given exponential ment into an equivalent logarithmic statement.

$10^{-2} = 0.01$ **log 0.01 = −2**

16. $10^3 = 1000$ **log 1000 = 3**

17. $10^{0.4771} = 3$ **log 3 = 0.4771**

18. $10^{7k} = r$ **$\log r = 7k$**

19. $e^{3.25} = 25.79$ **ln 25.79 = 3.25**

20. $e^{-4} = 0.0183$ **ln 0.0183 = −4**

21. $e^{\frac{12}{7}} = 5.5527$ **$\ln 5.5527 = \frac{12}{7}$**

22. $e^k = t$ **$\ln t = k$**

23. $e^{\frac{2}{r}} = w$ **$\ln w = \frac{2}{r}$**

24. $e^{4uv} = m$ **$\ln m = 4uv$**

In Exercises 25–36, evaluate the given expression without using a calculator.

25. $\log 10^{\sqrt{43}}$ **$\sqrt{43}$**

26. $\log 10^{\sqrt{x^2+y^2}}$ **$\sqrt{x^2 + y^2}$**

27. $\ln e^{15}$ **15**

28. $\ln e^{3.78}$ **3.78**

29. $\ln \sqrt{e}$ **$\frac{1}{2}$**

30. $\ln \sqrt[5]{e}$ **$\frac{1}{5}$**

31. $e^{\ln 931}$ **931**

32. $e^{\ln 34.17}$ **34.17**

33. $\ln e^{x+y}$ **$x + y$**

34. $\ln e^{x^2+2y}$ **$x^2 + 2y$**

35. $e^{\ln x^2}$ **x^2**

36. $e^{\ln \sqrt{x+3}}$ **$\sqrt{x + 3}$**

In Exercises 37–40, find the domain of the given function.

37. $f(x) = \ln(x + 1)$ **$(-1, \infty)$**

38. $g(x) = \ln(x + 2)$ **$(-2, \infty)$**

39. $h(x) = \log(-x)$ **$(-\infty, 0)$**

40. $k(x) = \log(2 - x)$ **$(-\infty, 2)$**

e ln 931

ADDITIONAL EXAMPLES

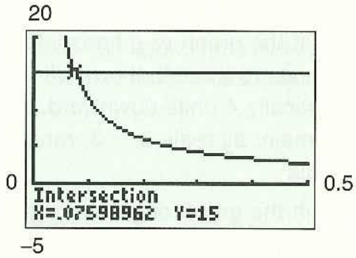
Example 7

If you invest money at an interest rate r, compounded annually, then $T(r)$ gives the time in years that it would take to triple.

$$T(r) = \frac{\ln 3}{\ln(1 + r)}$$

a. How long would it take to triple an investment of $500 at 5% annual interest? **approximately 22.5 years**

b. What annual interest rate is needed in order for the investment in part **a** to triple in 15 years? **7.60%**



Real-World Application

The function in **Example 7** is used to find the time in years it takes to double an investment at a certain interest rate. Accountants sometimes use an approximating technique called the *Rule of 72* for this task. The rule of 72 is: $f(R) = \frac{72}{R}$, where R is expressed as a percent. Have your students compare exact results with approximate results by entering $Y_1 = \frac{\ln 2}{\ln(1 + x)}$ and $Y_2 = \frac{72}{100x}$ into their calculators and using the TABLE feature, with increments of 0.01, to compare function values. (Note that the denominator of Y_2 is $100x$ because R is a percent, not a decimal.) The rule of 72 is also explored in exercise 65 (page 362).

X	Y1	Y2
.01	69.661	72
.02	35.003	36
.03	23.45	24
.04	17.673	18
.05	14.207	14.4
.06	11.896	12
.07	10.245	10.286

X=.07