a. *g* is the graph of *f* moved 3 units left.

b. *h* is the graph of *f* moved 2 units down.

c. *k* is the graph of *f* moved 3 units left, then 2 units down.

ction

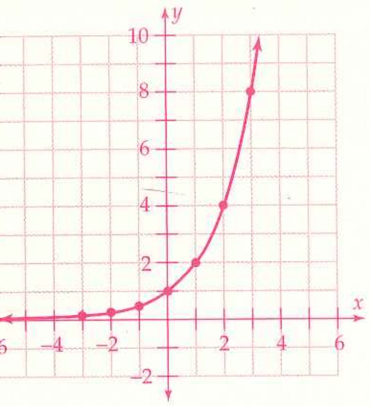
5.2 **Exponential Functions**

aching Notes

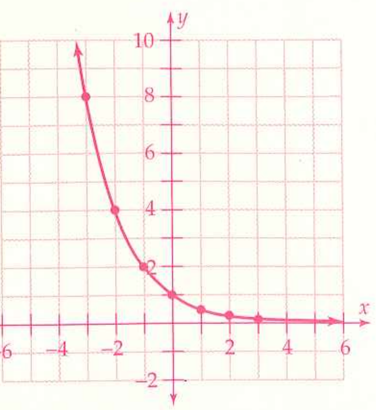
dents will better understand
perties of the **Graph of $f(x) = a^x$**
completing the following pencil-
-paper activity:

py and complete each table. Then
the points to graph each function.

	−3	−2	−1	0	1	2	3
x) = 2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



	−3	−2	−1	0	1	2	3
x) = $\left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$



100. Graph $f(x) = \sqrt{x}$ in the standard viewing window. Then, without doing any more graphing, describe the graphs of these functions.
a. $g(x) = \sqrt{x+3}$ *Hint:* $g(x) = f(x + 3)$; see Section 3.4.
b. $h(x) = \sqrt{x} - 2$
c. $k(x) = \sqrt{x+3} - 2$

101. Do Exercise 100 with $\sqrt[3]{\ }$ in place of $\sqrt{\ }$.

a. ***g* is the graph of *f* moved 3 units left.**

b. ***h* is the graph of *f* moved 2 units down.**

c. ***k* is the graph of *f* moved 3 units left, then 2 units down.**

5.2 Exponential Functions

Objectives

- Graph and identify transformations of exponential functions
- Use exponential functions to solve application problems

Graphs of Exponential Functions

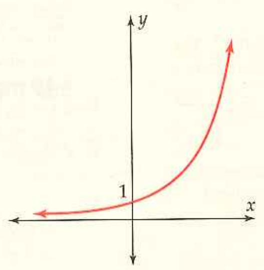
For each positive real number a, $a \neq 1$, there is an **exponential function with base *a*** whose domain is all real numbers and whose rule is $f(x) = a^x$. Some examples are shown below.

$$f(x) = 10^x \qquad g(x) = 2^x \qquad h(x) = \left(\frac{1}{2}\right)^x \qquad k(x) = \left(\frac{3}{2}\right)^x$$

The shape of the graph of an exponential function $f(x) = a^x$ depends only on the size of a, as shown in the following figures.

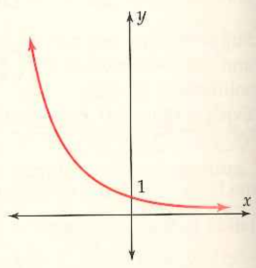
Graph of $f(x) = a^x$

$a > 1$



- **graph is above *x*-axis**
- ***y*-intercept is 1**
- **$f(x)$ is increasing**
- **$f(x)$ approaches the negative *x*-axis as *x* approaches $-\infty$**

$0 < a < 1$



- **graph is above *x*-axis**
- ***y*-intercept is 1**
- **$f(x)$ is decreasing**
- **$f(x)$ approaches the positive *x*-axis as *x* approaches ∞**

For $a = 0$ or $a = 1$, the function $f(x) = a^x$ is a constant function, not exponential. Even roots of negative numbers are not defined in the set of real numbers, so when $a < 0$, a^x is not defined for any rational exponent that

What is the *y*-intercept for each graph? 1
Will either graph go below the *x*-axis? no
Are both functions in the form $f(x) = a^x$? yes
For $g(x)$, is $a > 1$ or is $a < 1$? $a > 1$
Is $g(x)$ increasing or decreasing? increasing
What happens to $g(x)$ as x approaches $-\infty$?
It approaches the negative *x*-axis.
For $h(x)$, is $a > 1$ or is $a < 1$? $a < 1$
Is $h(x)$ increasing or decreasing? decreasing

What happens to $h(x)$ as x approaches ∞?
It approaches the positive *x*-axis.
Explain why $a \neq 1$ and $a \neq 0$:
If $a = 1$, $f(x) = 1^x = 1$.
If $a = 0$, $f(x) = 0^x = 0$ (for $x \neq 0$).
In either case, the graph is a horizontal line.

has an even number as its denominator. Because within any interval there are infinitely many rational numbers that have an even denominator, $f(x) = a^x$ has an infinite number of holes in every interval when $a < 0$. Therefore, the function is not *well-behaved* for $a < 0$, so it is not defined for those values.

The following two Graphing Explorations illustrate the effect that the value of a has on the shape of the graph of an exponential function for $a > 1$ and for $0 < a < 1$.

Graphing Exploration

a. Using a viewing window with $-3 \le x \le 7$ and $-2 \le y \le 18$, graph each function below on the same screen, and observe the behavior of each to the right of the y-axis.

$$f(x) = 1.3^x \quad g(x) = 2^x \quad h(x) = 10^x$$

As the graphs continue to the right, which graph rises least steeply? most steeply?

How does the steepness of the graph of $f(x) = a^x$ to the right of the y-axis seem to be related to the size of the base a?

b. Using the graphs of the same three functions in the viewing window with $-4 \le x \le 2$ and $-0.5 \le y \le 2$, observe the behavior to the *left* of the y-axis.

As the graph continues to the left, how does the size of the base a seem to be related to how quickly the graph of $f(x) = a^x$ falls toward the x-axis?

Graphing Exploration

Using a viewing window with $-4 \le x \le 4$ and $-1 \le y \le 4$, graph each function below on the same screen, and observe the behavior of each.

$$f(x) = 0.2^x \quad g(x) = 0.4^x \quad h(x) = 0.6^x \quad k(x) = 0.8^x$$

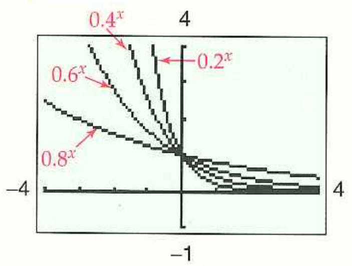
Notice that the bases of the exponential functions are increasing in size: $0 < 0.2 < 0.4 < 0.6 < 0.8 < 1$

As the graphs continue to the right, which graph falls least steeply? most steeply?

How does the steepness of the graph of $f(x) = a^x$ seem to be related to the size of the base a?

The graphing explorations above show that the graph of $f(x) = a^x$ rises or falls less steeply as the base a gets closer to 1.

Solution to second **Graphing Exploration:**



$k(x) = 0.8^x$ falls least steeply and $f(x) = 0.2^x$ falls most steeply. When $0 < a < 1$, steeper graphs correspond to smaller values of a.

Teaching Notes

The graph of $f(x) = a^x$ has an infinite number of holes within any interval when $a < 0$. *This is most likely a new and difficult concept for your students.* Try the following to help them understand.

Recall that even roots of negative numbers are not defined in the set of real numbers. For example, $(-16)^{\frac{3}{4}} = (\sqrt[4]{-16})^3$ is not defined.

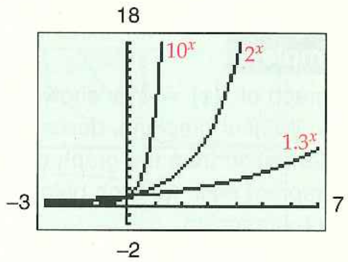
Now challenge students to choose any two rational numbers as x-values. Show that you can always find a number between the chosen numbers that has an even denominator (thereby making a^x undefined).

Example:

If $\frac{1}{5}$ and $\frac{1}{3}$ are chosen, show that $\frac{7}{30}$ is between them: $\frac{1}{5} = \frac{6}{30} < \frac{7}{30} < \frac{10}{30} = \frac{1}{3}$.

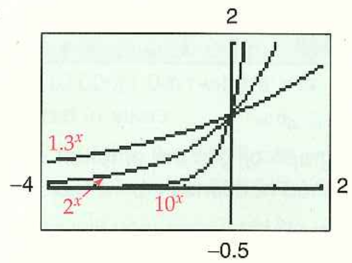
Solution to first **Graphing Exploration:**

a.



$f(x) = 1.3^x$ rises least steeply and $h(x) = 10^x$ rises most steeply. When $a > 1$, steeper graphs correspond to greater values of a.

b.



When $a > 1$, the graph falls toward the x-axis more quickly for greater values of a.

ADDITIONAL EXAMPLES

Example 1

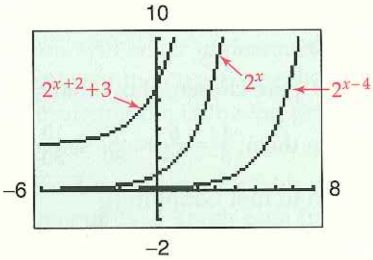
The graph of $f(x) = 2^x$ is shown in Figure 5.2-1. Without graphing, describe the transformation from the graph of f to the graph of each function below. Verify by graphing.

a. $g(x) = 2^{x-4}$

The graph of g is the graph of f shifted horizontally 4 units to the right. See graph below.

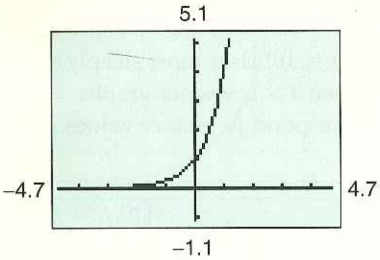
b. $h(x) = 2^{x+2} + 3$

The graph of h is the graph of f shifted horizontally 2 units to the left and vertically 3 units up.



Example 2

The graph of $f(x) = 4^x$ is shown below. Without graphing, describe the transformation from the graph of f to the graph of each function below. Verify by graphing.



$g(x) = 4^{0.5x}$

The graph of g is the graph of f stretched horizontally by a factor of 2.

$h(x) = 4^{0.4x}$

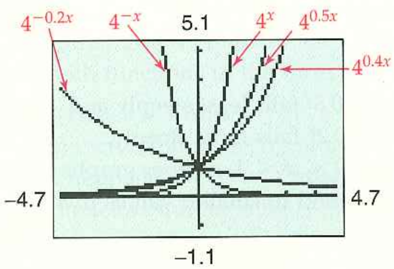
The graph of h is the graph of f stretched horizontally by a factor of 2.5.

$k(x) = 4^{-x}$

The graph of k is the graph of f reflected across the y-axis.

$p(x) = 4^{-0.2x}$

The graph of p is the graph of f stretched horizontally by a factor of 5 and reflected across the y-axis.



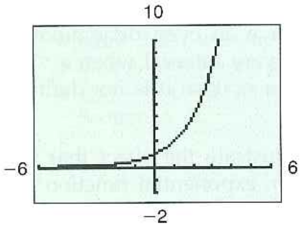


Figure 5.2-1

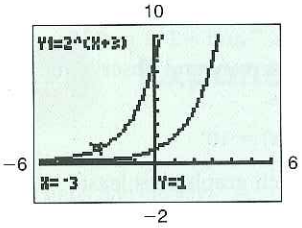


Figure 5.2-2

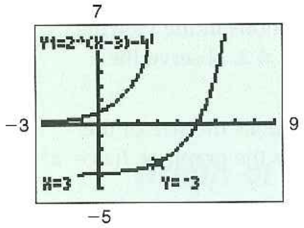


Figure 5.2-3

Example 1 **Translations**

The graph of $f(x) = 2^x$ is shown in Figure 5.2-1. Without graph describe the transformation from the graph of f to the graph of each fu tion below. Verify by graphing.

a. $g(x) = 2^{x+3}$ **b.** $h(x) = 2^{x-3} - 4$

Solution

a. If $f(x) = 2^x$, then $g(x) = 2^{x+3} = f(x + 3)$. So the graph of g is the graph of f shifted horizontally 3 units to the left, as shown in Figu 5.2-2.

b. If $f(x) = 2^x$, then $h(x) = 2^{x-3} - 4 = f(x - 3) - 4$. So the graph of $h(x)$ is the graph of $f(x) = 2^x$ shifted horizontally 3 units to the rig and vertically 4 units downward, as shown in Figure 5.2-3.

The graphs of exponential functions of the form $f(x) = a^x$ increase at a explosive rate. To see this, consider the graph of $f(x) = 2^x$ in Figure 5.2 If the x-axis were extended to the right, then $x = 50$ would be at the ri edge of the page. At this point, the graph of $f(x) = 2^x$ is 2^{50} units hi The scale of the y-axis in Figure 5.2-3 is about 12 units per inch, or 1 units per foot, or 760,320 units per mile. Therefore, the height of the gra at $x = 50$ is

$$\frac{2^{50}}{760{,}320} = 1{,}480{,}823{,}741 \text{ miles},$$

which would put that part of the graph well beyond the planet Satur

Since most quantities that grow exponentially do not change as dram ically as the graph of $f(x) = 2^x$, exponential functions that model real-l growth or decay are usually modified by the insertion of appropriate co stants. These functions are generally of the form

$$f(x) = Pa^{kx},$$

such as the functions shown below.

$$f(x) = \frac{1}{2}(5.2^{0.45x}) \qquad g(x) = 3.5(10^{-0.03x}) \qquad h(x) = (-6)(1.076^{2x})$$

Their graphs have the same shape as the graph of $f(x) = a^x$, but may ri or fall more or less steeply, depending on the constants P, k, and a.

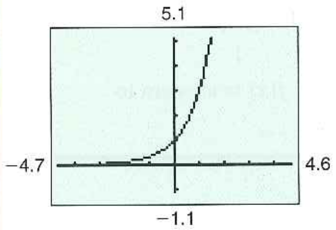


Figure 5.2-4

Example 2 **Horizontal Stretches**

The graph of $f(x) = 3^x$ is shown in Figure 5.2-4. Without graphi describe the transformation from the graph of f to the graph of each fu tion below. Verify by graphing.

$$g(x) = 3^{0.2x} \qquad h(x) = 3^{0.8x} \qquad k(x) = 3^{-x} \qquad p(x) = 3^{-0.4x}$$

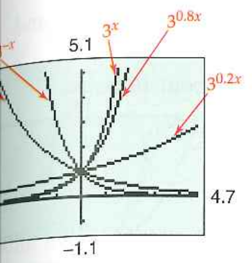


Figure 5.2-5

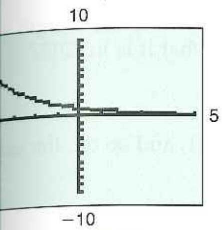


Figure 5.2-6

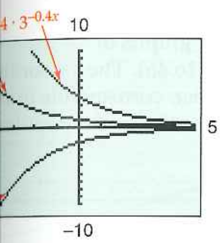


Figure 5.2-7

Solution

The graphs of $g(x) = 3^{0.2x}$ and $h(x) = 3^{0.8x}$ are the graph of f stretched horizontally by a factor of $\frac{1}{0.2} = 5$ and $\frac{1}{0.8} = 1.25$, respectively. The graph of $k(x) = 3^{-x}$ is the graph of f reflected across the y-axis. The graph of $p(x) = 3^{-0.4x}$ is the graph of f stretched horizontally by a factor of $\frac{1}{0.4} = 2.5$ and reflected across the y-axis. The graphs are identified in Figure 5.2.5.

Example 3 Vertical Stretches

The graph of $p(x) = 3^{-0.4x}$ is shown in Figure 5.2-6. Without graphing, describe the transformation from the graph of p to the graph of each function below. Verify by graphing.

$$q(x) = 4 \cdot 3^{-0.4x} \qquad r(x) = (-2)3^{-0.4x}$$

Solution

The graph of $q(x) = 4 \cdot 3^{-0.4x}$ is the graph of $p(x) = 3^{-0.4x}$ stretched vertically by a factor of 4. The graph of $r(x) = (-2)3^{-0.4x}$ is the graph of $p(x) = 3^{-0.4x}$ stretched vertically by a factor of 2 and reflected across the x-axis. The graphs are identified in Figure 5.2-7.

Exponential Growth and Decay

In this section, you will see that exponential functions are useful for modeling situations in which a quantity increases or decreases by a fixed factor. In Section 5.3 you will learn how to construct these types of functions.

Example 4 Finance

If you invest \$5000 in a stock that is increasing in value at the rate of 3% per year, then the value of your stock is given by the function $f(x) = 5000(1.03)^x$, where x is measured in years.

a. Assuming that the value of your stock continues growing at this rate, how much will your investment be worth in 4 years?

b. When will your investment be worth \$8000?

Solution

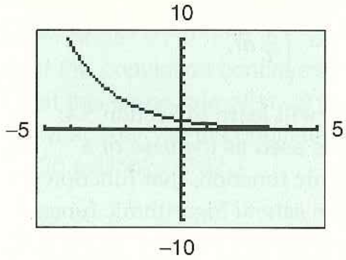
a. Letting $x = 4$, $f(4) = 5000(1.03)^4 \approx 5627.54$.
In 4 years your stock is worth about \$5627.54.

b. Find the value of x for which $f(x) = 8000$. In other words, solve the equation $5000(1.03)^x = 8000$.

ADDITIONAL EXAMPLES

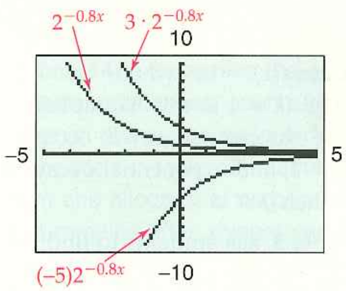
Example 3

The graph of $p(x) = 2^{-0.8x}$ is shown below. Without graphing, describe the transformation from the graph of p to the graph of each function below. Verify by graphing.



$q(x) = 3 \cdot 2^{-0.8x}$
The graph of q is the graph of p stretched vertically by a factor of 3.

$r(x) = (-5)2^{-0.8x}$
The graph of r is the graph of p stretched vertically by a factor of 5 and reflected across the x-axis.

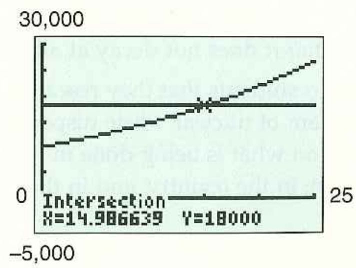


Example 4

If you invest \$10,000 in a certificate of deposit with an interest rate of 4% per year, the value of your certificate is given by the function $f(x) = 10{,}000(1.04)^x$, where x is measured in years.

a. How much is the certificate worth in 5 years? about \$12,166.53

b. When will your certificate be worth \$18,000? about 15 years



Math Background

he irrational number e has several
fferent, equivalent definitions:

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

is the number such that

$$e = 1 = \int_1^e \frac{1}{t}\,dt.$$

udents will learn in Section 5.4:
/hen e is used as the base of a
garithmic function, that function is
alled the *natural* logarithmic function.
he exponential function $f(x) = e^x$
nd the natural logarithmic function
$x) = \ln x$ are inverses.

xample Notes

xamples 4, 5, and 7 are examples
f exponential growth. Each function
of the form $f(x) = ka^x$, where $a > 1$.
xample 6 is an example of
xponential decay, with a function
f the form $f(x) = ka^x$, where
$< a < 1$. ($k = 1$ in this Example.)
Jote that exponential growth occurs
/hen $a > 1$, and exponential decay
ccurs when $0 < a < 1$.

n Example 5, ask students to find the
alue of $g(0)$ without using a
alculator, and to interpret the result.
$(0) = 2.5(1.0185)^0 = 2.5(1) = 2.5$
The world population in 1950 was 2.5
illion.

Real-World Application

The topic of Example 6, radioactive
lecay, should concern all citizens, not
ust scientists and government
fficials. As noted in the Example,
lutonium decays so slowly that, for
ractical considerations, we can
ssume that it does not decay at all.
uggest to students that they research
he problem of nuclear waste disposal,
ocusing on what is being done in
heir state, in the country, and in the
vorld.

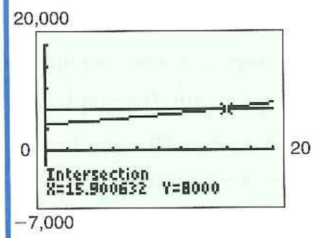


Figure 5.2-8

The point of intersection of the graphs of $f(x) = 5000(1.03)^x$ and $y = 8000$ is approximately (15.901, 8000).

Therefore, the stock will be worth \$8000 in about 16 years.

Example 5 Population Growth

Based on data from the past 50 years, the world population, in billions, can be approximated by the function $g(x) = 2.5(1.0185)^x$, where $x = 0$ corresponds to 1950.

a. Estimate the world population in 2015.

b. In what year will the population be double what it is in 2015?

Solution

a. Since $x = 0$ corresponds to 1950, $x = 1$ to 1951, and so on, the year 2015 corresponds to $x = 65$. Find g(65).

$$g(65) = 2.5(1.0185)^{65} \approx 8.23$$

The world population in 2015 will be about 8.23 billion people.

b. Twice the population in 2015 is $2(8.23) = 16.46$ billion. Find the number x such that $g(x) = 16.46$; that is, solve $2.5(1.0185)^x = 16.46$.

A graphical intersection finder shows that the approximate coordinates of the point of intersection of the graphs of $g(x) = 2.5(1.0185)^x$ and $y = 16.46$ are (102.81, 16.46). The x-coordinate 102.81, or 103 when rounded to the nearest year, corresponds to the year 2053. Notice that it takes only 38 years for the world population to double.

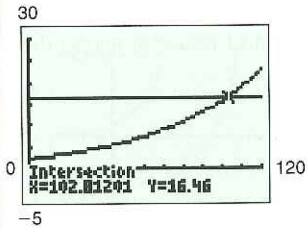


Figure 5.2-9

Example 6 Radioactive Decay

The amount from one kilogram of plutonium (^{239}Pu) that remains after x years can be approximated by the function $M(x) = 0.99997^x$. Estimate the amount of plutonium remaining after 10,000 years.

Solution

Because M is an exponential function with a base smaller than 1 but very close to 1, its graph falls very slowly from left to right. The fact that the graph falls so slowly as x gets large means that even after an extremely long time, a substantial amount of plutonium will remain.

When $x = 10{,}000$, $M(x) \approx 0.74$. Therefore, almost three-fourths of the original plutonium remains after 10,000 years! This is the reason that nuclear waste disposal is such a serious concern.

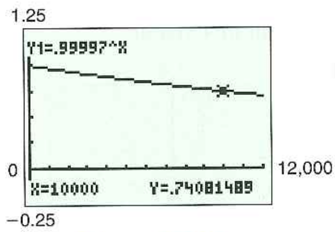


Figure 5.2-10

The Number *e* and the Natural Exponential Function

There is an irrational number, denoted e, that arises naturally in a variety of phenomena and plays a central role in the mathematical description of the physical universe. Its decimal expansion begins as shown below.

$$e = 2.718281828459045\ldots$$

Most calculators have an e^x key that can be used to evaluate the natural exponential function $f(x) = e^x$. When you evaluate e^1 using a calculator, the display will show the first part of the decimal expansion of e.

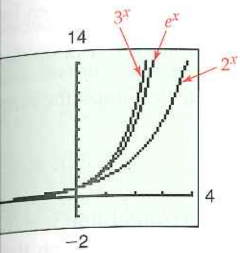


Figure 5.2-11

Figure 5.2-11 shows that the graph of $f(x) = e^x$ has the same shape as the graphs of $y = 2^x$ and $y = 3^x$, but it climbs more steeply than the graph of $y = 2^x$ and less steeply than the graph of $y = 3^x$.

Example 7 Population Growth

If the population of the United States continues to grow as it has since 1980, then the approximate population, in millions, of the United States in year t, where $t = 0$ corresponds to the year 1980, will be given by the function $P(t) = 227e^{0.0093t}$.

a. Estimate the population in 2015.

b. When will the population reach half a billion?

Solution

a. The year 2015 corresponds to $t = 35$. Find $P(35)$.

$$P(35) = 227e^{0.0093(35)} \approx 314.3$$

Therefore, the population in 2015 will be approximately 314.3 million people.

b. Half a billion is 500 million. Find the value of t for which $P(t) = 500$. A graphical intersection finder shows that the approximate coordinates of the point of intersection of the graphs of $P(t) = 227e^{0.0093t}$ and $y = 500$ are approximately (85, 500). A t-value of 85 corresponds to the year 2065. Therefore, the population will reach half a billion approximately by the year 2065.

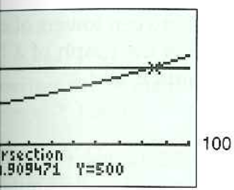


Figure 5.2-12

Other Exponential Functions

In most real-world applications, populations cannot grow infinitely large. The population growth models shown previously do not take into account factors that may limit population growth in the future. Example 8 illustrates a function, called a **logistic model**, which is designed to model situations that have limited future growth due to a fixed area, food supply, or other factors.

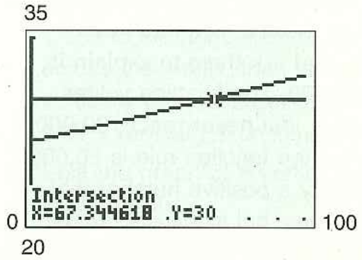
ADDITIONAL EXAMPLES

Example 5

The projected population of Tokyo, Japan, in millions, from 2000 to 2015 can be approximated by the function $g(x) = 26.4(1.0019)^x$, where $x = 0$ corresponds to the year 2000.

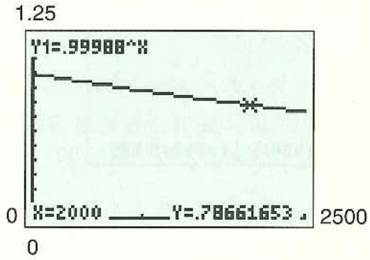
a. Estimate the population of Tokyo in 2015. 27.16 million

b. If the population continues to grow at this same rate after 2015, in what year will the population reach 30 million? 2067



Example 6

Geologists and archeologists use carbon-14 to determine the age of organic substances, such as bones or small plants and animals found embedded in rocks. The amount from one kilogram of carbon-14 that remains after x years can be approximated by the function $M(x) = 0.99988^x$. Estimate the amount of carbon-14 remaining after 2000 years. about 0.79 kg

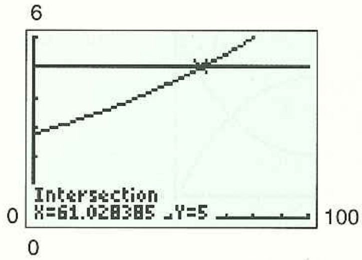


ample 7

population of Los Angeles, California, e 1970, in millions, can be modeled by function $P(t) = 2.7831e^{0.0096t}$, where t esents time in years. Assuming that this el continues to be appropriate, use it to wer the following:

stimate the population in 2010. 4.1 million

b. When will the population reach 5 million? in 2031



Example Notes

For **Example 8,** the following will help students understand that the function $p(t)$ approaches, but never quite reaches, 20,000:

Have students evaluate $24e^{-\frac{t}{4}}$ for $t = 0, 1, 10,$ and 40 (rounding the first three to the nearest whole number). 24, 19, 2, 0.001 This will help them see that, as t increases, the value of

$$p(t) = \frac{20{,}000}{1 + 24e^{-\frac{t}{4}}} \text{ approaches } \frac{20{,}000}{1 + 0}, \text{ or}$$

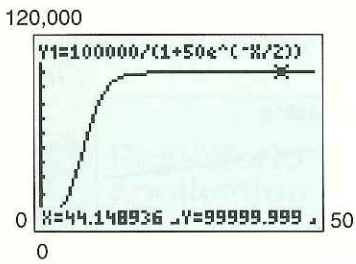
20,000.

Ask students for the equation of the horizontal asymptote in the graph, and ask them to explain it. $y = 20{,}000$; The function values approach, but never reach, 20,000 because the function rule is 20,000 divided by a positive number that approaches, but is always greater than, 1.

ADDITIONAL EXAMPLES

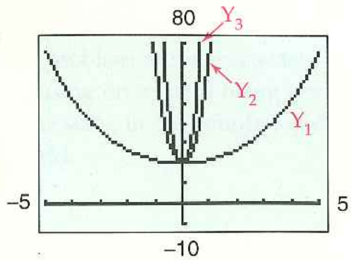
Example 8

The population of certain bacteria in a beaker at time t hours is given by the function $p(t) = \frac{100{,}000}{1 + 50e^{-\frac{t}{2}}}$, where $t \geq 0$. Graph the function and find the upper limit on the bacteria population. 100,000



Teaching Notes

Solution to the **Graphing Exploration:** Greater x-coefficients correspond to steeper and narrower graphs.



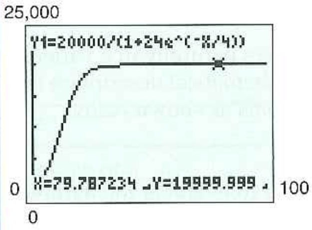


Figure 5.2-13

Example 8 Logistic Model

The population of fish in a certain lake at time t months is given by the function $p(t) = \frac{20{,}000}{1 + 24e^{-\frac{t}{4}}}$, where $t \geq 0$. There is an upper limit on the fish population due to the oxygen supply, available food, etc. Graph the function and find the upper limit on the fish population.

Solution

The graph of $p(t)$ at the left suggests that the horizontal line $y = 20{,}000$ is a horizontal asymptote of the graph. If so, the upper limit on the fish population is 20,000.

You can verify this by rewriting the rule of p as shown below.

$$p(t) = \frac{20{,}000}{1 + 24e^{-\frac{t}{4}}} = \frac{20{,}000}{1 + \frac{24}{e^{\frac{t}{4}}}}$$

As t increases, $\frac{t}{4}$ increases and $e^{\frac{t}{4}}$ grows very large. As $e^{\frac{t}{4}}$ grows very large, $\frac{24}{e^{\frac{t}{4}}}$ gets very close to 0. As $\frac{24}{e^{\frac{t}{4}}}$ gets closer and closer to 0, $p(t)$ gets closer and closer to $\frac{20{,}000}{1 + 0}$, or 20,000. Because $e^{\frac{t}{4}}$ is positive and $\frac{24}{e^{\frac{t}{4}}}$ never quite reaches 0, the denominator of $p(t)$ is always slightly larger than 1 and $p(t)$ is always less than 20,000.

When a cable, such as a power line, is suspended between towers of equal height, it forms a curve called a catenary, which is the graph of a function of the form shown below for suitable constants A and k.

$$f(x) = A(e^{kx} + e^{-kx})$$

The Gateway Arch in St. Louis, shown in Figure 5.2-14, has the shape of an inverted catenary, which was chosen because it evenly distributes the internal structural forces.

Figure 5.2-14

Graphing Exploration

Using the viewing window with $-5 \leq x \leq 5$ and $-10 \leq y \leq 80$, graph each function below on the same screen, and observe their behavior.

$Y_1 = 10(e^{0.4x} + e^{-0.4x})$ $Y_2 = 10(e^{2x} + e^{-2x})$ $Y_3 = 10(e^{3x} + e^{-3x})$

How does the coefficient of x affect the shape of the graph? Predict the shape of the graph of $y = -Y_1 + 80$. Confirm your answer by graphing.

The graph of $-Y_1 + 80$ will be the graph of Y_1 reflected across the x-axis and moved up 80 units.