apter Outline

Interdependence of Sections

$5.1 \longrightarrow 5.2 \longrightarrow 5.4 \longrightarrow 5.5$; $5.2 \nearrow 5.3$; $5.5 \nearrow 5.6$; $5.5 \searrow 5.7$

Section 5.1 contains prerequisite review material for this chapter. If students are familiar enough with the objectives of this section, it may be skipped.

Exponential and logarithmic functions are essential for the mathematical description of a variety of phenomena in the physical sciences, engineering, and economics. Although a calculator is necessary to evaluate these functions for most values, you will not be able to use your calculator efficiently or interpret its answers unless you understand the properties of these functions. When calculations can readily be done by hand, you will be expected to do them without a calculator.

.1 Radicals and Rational Exponents

ectives

- efine and apply rational nd irrational exponents
- implify expressions ontaining radicals or tional exponents

OTE All constants, ables, and solutions in chapter are real bers.

*n*th Roots

Recall that when $c \geq 0$, the square root of c is the nonnegative solution of the equation $x^2 = c$. Cube roots, fourth roots, and higher roots are defined in a similar fashion as solutions of the equation $x^n = c$.

This equation can be solved graphically by finding the x-coordinate of the intersection points of the graphs of $y = x^n$ and $y = c$. (Review finding solutions graphically in Section 2.1 and the shape of the graph of $y = ax^n$ in Section 4.3, if needed.)

Depending on whether n is even or odd and whether c is positive or negative, $x^n = c$ may have two, one, or no solutions, as shown in the following figures.

Section 5.1 Radicals and Rational Exponents

Teaching Notes

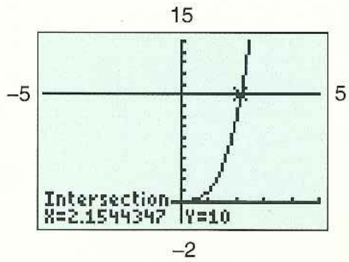
Students have most recently studied the complex numbers. Tell them that now we are back on more familiar ground, the real numbers.

Tell students that ***n*th Roots** will be defined using the equation $x^n = c$. If $n = 3$ and $c = 10$, then $x^n = c$ becomes $x^3 = 10$. Students can solve $x^3 = 10$ graphically, as shown, using $Y_1 = x^3$ and $Y_2 = 10$.

Ask students how they can verify that the solution is reasonable.

$2^3 = 8$, so the solution of $x^3 = 10$ should be a little more than 2, and 2.1544347 is a little more than 2.

Encourage them to make a habit of verifying whether solutions are reasonable, using known powers of integers.



Solutions of $x^n = c$			
n odd	n even		
Exactly one solution for any c	$c > 0$ One positive and one negative solution	$c = 0$ One solution $x = 0$	$c < 0$ No solution

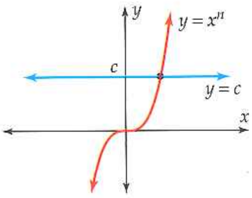
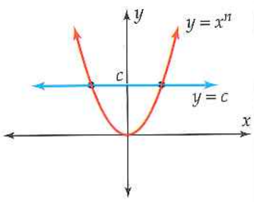
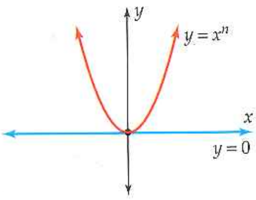
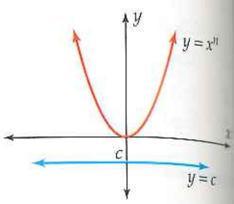

			

Figure 5.1-1	Figure 5.1-2	Figure 5.1-3	Figure 5.1-4

The figures illustrate the following definition of nth roots.

*n*th Roots

Let c be a real number and n a positive integer. The *nth root of c* is denoted by either of the symbols

$$\sqrt[n]{c} \quad \text{or} \quad c^{\frac{1}{n}}$$

and is defined to be

- **the solution of $x^n = c$ when n is odd; or**
- **the nonnegative solution of $x^n = c$ when n is even and $c \geq 0$.**

Examples of nth roots are shown below.

$\sqrt[3]{-8} = (-8)^{\frac{1}{3}} = -2$ because -2 is the solution of $x^3 = -8$.
$\sqrt[4]{81} = (81)^{\frac{1}{4}} = 3$ because 3 is the nonnegative solution of $x^4 = 81$.

Expressions involving nth roots can often be simplified or written in a variety of ways by using a basic fact of exponents.

$$\sqrt[n]{cd} = \sqrt[n]{c}\sqrt[n]{d} \qquad \text{or equivalently,} \qquad (cd)^{\frac{1}{n}} = c^{\frac{1}{n}}d^{\frac{1}{n}}$$

Example 1 Operations on *n*th Roots

Simplify each expression.

a. $\sqrt{8} \cdot \sqrt{12}$ **b.** $\sqrt{12} - \sqrt{75}$
c. $\sqrt[3]{8x^6y^4}$ **d.** $(5 + \sqrt{c})(5 - \sqrt{c})$, where $c > 0$

ching Notes

y be worthwhile for students to some equations graphically with lators to illustrate the first two es at the right. For Figure 5.1-1, est $x^5 = 30$ (graph $y = x^5$ and 0) and $x^3 = -10$ (graph $y = x^3$ $= -10$). For Figure 5.1-2, they olve $x^2 = 20$ (graph $y = x^2$ and 20) and $x^4 = 80$ (graph $y = x^4$ and 80). Again, have students verify heir solutions are reasonable.

that $y = x$ belongs to the group of ions represented by Figure 5.1-1, use $x = x^1$. Students should e that, although the graph of is a line instead of S-shaped, is exactly one solution of $x^1 = c$ ny c.

he definition of *n*th Roots, show ents the following equivalent essions, which are probably iar to them:

and $8^{\frac{1}{3}}$
and $25^{\frac{1}{2}}$

generalize, saying that $\sqrt[n]{c}$ and e equivalent for any real number d positive integer n.

ubstituting $\sqrt[n]{c}$ for x in the tion $x^n = c$, we have the wing facts:

odd n, $(\sqrt[n]{c})^n = c$.
even n and $c \geq 0$, $(\sqrt[n]{c})^n = c$.

numerical examples below trate various cases of nth roots, essing odd and even values of d signs of c:

$= 2$, $\sqrt[3]{-8} = -2$
$= 2$, $\sqrt[4]{-16}$ is not defined.
$= 0$, $\sqrt[4]{0} = 0$

COMMON ERROR ALERT

lents might try to apply the special uct $(a + b)(a - b) = a^2 - b^2$ n expression such as $\sqrt{c})(2 - \sqrt{c})$, getting $10 - c$ he answer. Point out that in $\sqrt{c})(2 - \sqrt{c})$, $5 \neq 2$.

CAU
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225^
225^

Solution

a. $\sqrt{8} \cdot \sqrt{12} = \sqrt{8 \cdot 12} = \sqrt{96} = \sqrt{16 \cdot 6} = \sqrt{16} \cdot \sqrt{6} = 4\sqrt{6}$

b. $\sqrt{12} - \sqrt{75} = \sqrt{4 \cdot 3} - \sqrt{25 \cdot 3} = \sqrt{4}\sqrt{3} - \sqrt{25}\sqrt{3}$
$= 2\sqrt{3} - 5\sqrt{3} = -3\sqrt{3}$

c. $\sqrt[3]{8x^6y^4} = \sqrt[3]{8} \cdot \sqrt[3]{x^3 \cdot x^3} \cdot \sqrt[3]{y^3 \cdot y} = 2x^2y\sqrt[3]{y}$

d. $(5 + \sqrt{c})(5 - \sqrt{c}) = 5^2 - (\sqrt{c})^2 = 25 - c$

When using a calculator, exponent notation for nth roots is usually preferred over radical notation.

AUTION

hen using exponent otation to evaluate nth ots with a calculator, sure to use parenthes when raising to the actional power. ample: To enter $\sqrt[3]{9}$, ess 9^(1/3).

Example 2 Evaluating *n*th Roots

Use a calculator to approximate each expression to the nearest ten-thousandth.

a. $40^{\frac{1}{5}}$ **b.** $225^{\frac{1}{11}}$

Solution

a. Because $\frac{1}{5} = 0.2$, the expressions $40^{\frac{1}{5}}$ and $40^{0.2}$ are equivalent, as shown at right.

$$40^{\frac{1}{5}} \approx 2.0913$$

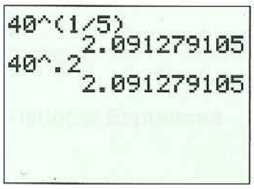

Figure 5.1-5

b. The fraction $\frac{1}{11}$ is equivalent to the repeating decimal $0.090909\cdots$. The fraction $\frac{1}{11}$ is not equivalent to this decimal if it is rounded off, as shown at left. Therefore, it is better to leave the exponent in fractional form.

$$225^{\frac{1}{11}} \approx 1.6362$$

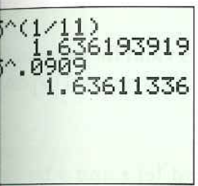

Figure 5.1-6

Rational Exponents

Rational exponents of the form $\frac{1}{n}$ are called nth roots. Rational exponents can also be of the form $\frac{m}{n}$, such as $4^{\frac{3}{2}}$. Rational exponents of the form $\frac{m}{n}$ can be defined in such a way that the laws of exponents, such as $c^{rs} = (c^r)^s$, are still valid. For example, because $\frac{3}{2} = (3)\frac{1}{2} = \left(\frac{1}{2}\right)3$, it is reasonable to say that $4^{\frac{3}{2}} = (4^3)^{\frac{1}{2}} = (4^{\frac{1}{2}})^3$. These expressions are equivalent.

$$4^{\frac{3}{2}} = (4^3)^{\frac{1}{2}} = (64)^{\frac{1}{2}} = \sqrt{64} = 8$$
$$4^{\frac{3}{2}} = (4^{\frac{1}{2}})^3 = (\sqrt{4})^3 = (2)^3 = 8$$

This illustrates the definition of rational exponents.

Example Notes

An alternate method for **Example 1a** is:
$\sqrt{8} \cdot \sqrt{12} = 2\sqrt{2} \cdot 2\sqrt{3} = 4\sqrt{6}$

In **Example 2,** direct students' attention to the **CAUTION** box. Have them enter 40^1/5 into their calculator. Ask them to explain why they get 8. The calculator interprets this as 40^1 divided by 5.

COMMON ERROR ALERT

A common error is to only partially simplify a radical expression, using a perfect square factor in the radicand, but not using the *greatest* perfect square factor. For example,
$\sqrt{48} = \sqrt{4 \cdot 12} = 2\sqrt{12}$, but
$2\sqrt{12}$ is not in simplest form because 12 contains a perfect square factor. If students use the factorization above, they must factor again to completely simplify the expression:
$\sqrt{48} = \sqrt{4 \cdot 12} = 2\sqrt{12}$
$= 2\sqrt{4 \cdot 3} = 2 \cdot 2\sqrt{3} = 4\sqrt{3}$

ADDITIONAL EXAMPLES

Example 1

Simplify each expression.

a. $\sqrt{6} \cdot \sqrt{8}$ $4\sqrt{3}$

b. $\sqrt{45} - \sqrt{125}$ $-2\sqrt{5}$

c. $\sqrt[3]{1000x^3y^7}$ $10xy^2\sqrt[3]{y}$

d. $(6 + \sqrt{b})(6 - \sqrt{b})$, where $b > 0$
$36 - b$

Example 2

Use a calculator to approximate each expression to the nearest ten-thousandths.

a. $35^{\frac{1}{8}}$ ≈ 1.5596

b. $285^{\frac{1}{12}}$ ≈ 1.6017

aching Notes

e that in the **Definition of onal Exponents,** k must be tive. If k were allowed to be negative, then meaningless ressions of the forms $\sqrt[0]{c}$ and $\sqrt[-3]{c}$ uld occur.

expression $c^{\frac{t}{k}}$ is defined in both onential notation and radical ation. Make sure that students erstand which radical notation is ivalent to each exponential ation:

$(c^t)^{\frac{1}{k}} = \sqrt[k]{c^t}$

$(c^{\frac{1}{k}})^t = (\sqrt[k]{c})^t$

demonstrate the value of being nfortable with both radical forms, students to evaluate $\sqrt[3]{125^2}$ hout a calculator. Then show

$25^2 = (\sqrt[3]{125})^2 = (5)^2 = 25.$

phasize that the **Laws of onents** are now stated for *rational* onents. Use numerical examples h as these to illustrate some of laws:

$8^{\frac{2}{3}} \cdot 8^{\frac{1}{3}} = 4 \cdot 2 = 8$

$8^{\frac{2}{3}} \cdot 8^{\frac{1}{3}} = 8^{\frac{2}{3}+\frac{1}{3}} = 8^1 = 8$

$\frac{8^{\frac{2}{3}}}{8^{\frac{1}{3}}} = \frac{4}{2} = 2$

$\frac{8^{\frac{2}{3}}}{8^{\frac{1}{3}}} = 8^{\frac{2}{3}-\frac{1}{3}} = 8^{\frac{1}{3}} = 2$

$(8 \cdot 27)^{\frac{2}{3}} = (216)^{\frac{2}{3}}$
$= (\sqrt[3]{216})^2$
$= (6)^2$
$= 36$

$(8 \cdot 27)^{\frac{2}{3}} = 8^{\frac{2}{3}} \cdot 27^{\frac{2}{3}} = 4 \cdot 9 = 36$

$\left(\frac{8}{27}\right)^{\frac{2}{3}} = \left(\sqrt[3]{\frac{8}{27}}\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$

$\left(\frac{8}{27}\right)^{\frac{2}{3}} = \frac{8^{\frac{2}{3}}}{27^{\frac{2}{3}}} = \frac{4}{9}$

Definition of Rational Exponents

Let c be a positive real number and let $\frac{t}{k}$ be a rational number with positive denominator.

$c^{\frac{t}{k}}$ is defined to be the number $(c^t)^{\frac{1}{k}} = (c^{\frac{1}{k}})^t$

In radical notation, $c^{\frac{t}{k}} = \sqrt[k]{c^t} = (\sqrt[k]{c})^t$.

Every terminating decimal is a rational number; therefore, expressions such as $13^{3.78}$ can be expressed as $13^{\frac{378}{100}}$. Although the definition of rational exponents requires c to be positive, it remains valid when c is negative, *provided that* the exponent is in lowest terms with an odd denominator, such as $(-8)^{\frac{2}{3}}$. In Exercise 89, you will explore why these restrictions are necessary when c is negative.

CAUTION

Although $(-8)^{\frac{2}{3}} = \sqrt[3]{(-8)^2} = \sqrt[3]{64} = 4$, and 4 is a real number, entering $(-8)^{\frac{2}{3}}$ on some calculators may produce either an error message or a complex number. If this occurs, you can get the correct answer by entering one of the equivalent expressions below.

$[(-8)^2]^{\frac{1}{3}}$ or $[(-8)^{\frac{1}{3}}]^2$

Laws of Exponents

You have seen that the law of exponents $c^{rs} = (c^r)^s$ is valid for rational exponents. In fact, all of the laws of exponents are valid for rational exponents.

Laws of Exponents

Let c and d be nonnegative real numbers and let r and s be rational numbers. Then

1. $c^r c^s = c^{r+s}$

2. $\frac{c^r}{c^s} = c^{r-s}$ $(c \neq 0)$

3. $(c^r)^s = c^{rs}$

4. $(cd)^r = c^r d^r$

5. $\left(\frac{c}{d}\right)^r = \frac{c^r}{d^r}$ $(d \neq 0)$

6. $c^{-r} = \frac{1}{c^r}$ $(c \neq 0)$

If $c \neq 1$ and $d \neq 1$,

- **$c^r = c^s$ if and only if $r = s$.**
- **$c^r = d^r$ if and only if $c = d$.**

Example 3 Simplifying Expressions with Rational Exponents

Write the expression $(8r^{\frac{3}{4}}s^{-3})^{\frac{2}{3}}$ using only positive exponents.

Solution

$$\begin{aligned}(8r^{\frac{3}{4}}s^{-3})^{\frac{2}{3}} &= 8^{\frac{2}{3}}(r^{\frac{3}{4}})^{\frac{2}{3}}(s^{-3})^{\frac{2}{3}} && (cd)^r = c^r d^r\\ &= \sqrt[3]{8^2}\,(r^{\frac{6}{12}})(s^{-\frac{6}{3}}) && \text{definition and } (c^r)^s = c^{rs}\\ &= \sqrt[3]{64}\,(r^{\frac{1}{2}})(s^{-2}) && \text{simplify}\\ &= \frac{4r^{\frac{1}{2}}}{s^2} && \text{simplify and } c^{-r} = \frac{1}{c^r}\end{aligned}$$

The expression $\frac{4r^{\frac{1}{2}}}{s^2}$ can also be written as $= \frac{4\sqrt{r}}{s^2}$ if it is more convenient.

Example 4 Simplifying Expressions with Rational Exponents

Simplify the expression $x^{\frac{1}{2}}(x^{\frac{3}{4}} - x^{\frac{3}{2}})$.

Solution

$$\begin{aligned}x^{\frac{1}{2}}(x^{\frac{3}{4}} - x^{\frac{3}{2}}) &= x^{\frac{1}{2}}x^{\frac{3}{4}} - x^{\frac{1}{2}}x^{\frac{3}{2}} && a(b-c) = ab - ac\\ &= x^{\frac{1}{2}+\frac{3}{4}} - x^{\frac{1}{2}+\frac{3}{2}} && c^r c^s = c^{r+s}\\ &= x^{\frac{5}{4}} - x^{\frac{4}{2}}\\ &= x^{\frac{5}{4}} - x^2\end{aligned}$$

Example 5 Simplifying Expressions with Rational Exponents

Simplify the expression $(x^{\frac{5}{2}}y^4)(xy^{\frac{7}{4}})^{-2}$.

Solution

$$\begin{aligned}(x^{\frac{5}{2}}y^4)(xy^{\frac{7}{4}})^{-2} &= (x^{\frac{5}{2}}y^4)(x^{-2})(y^{\frac{7}{4}})^{-2} && (cd)^r\\ &= (x^{\frac{5}{2}}y^4)(x^{-2})(y^{-\frac{14}{4}}) && (c^r)^s\\ &= x^{\frac{5}{2}}x^{-2}y^4y^{-\frac{7}{2}} && \text{commutative}\\ &= x^{\frac{5}{2}-2}y^{4-\frac{7}{2}} && c^r c^s = c^{r+s}\\ &= x^{\frac{1}{2}}y^{\frac{1}{2}} && \text{simplify}\end{aligned}$$

Example 6 Simplifying Expressions with Rational Exponents

Let k be a positive rational number. Write the expression $\sqrt[10]{c^{5k}}\sqrt{(c^{-k})^{\frac{1}{2}}}$ without radicals, using only positive exponents.

Solution

$$\begin{aligned}\sqrt[10]{c^{5k}}\sqrt{(c^{-k})^{\frac{1}{2}}} &= (c^{5k})^{\frac{1}{10}}[(c^{-k})^{\frac{1}{2}}]^{\frac{1}{2}} && \text{definition}\\ &= c^{\frac{5k}{10}}(c^{-\frac{k}{2}})^{\frac{1}{2}} && (c^r)^s = c^{rs}\\ &= c^{\frac{k}{2}}c^{-\frac{k}{4}} && \text{simplify and } (c^r)^s = c^{rs}\\ &= c^{\frac{k}{2}-\frac{k}{4}} && c^r c^s = c^{r+s}\\ &= c^{\frac{k}{4}} && \text{simplify}\end{aligned}$$

Example Notes

While discussing **Example 3** through **Example 6**, you may want to review operations with fractions. Remind students that a common denominator is needed to add or subtract, but not to multiply or divide. The following are from **Example 5**:

To find $\left(y^{\frac{7}{4}}\right)^{-2}$, multiply $\frac{7}{4}\cdot(-2)$.

$\frac{7}{4}\cdot(-2) = \frac{7}{4}\cdot\frac{-2}{1} = \frac{-14}{4} = -\frac{7}{2}$

To find $y^4 \cdot y^{-\frac{7}{2}}$, add $4 + \left(-\frac{7}{2}\right)$.

$4 + \left(-\frac{7}{2}\right) = \frac{8}{2} + \left(-\frac{7}{2}\right) = \frac{1}{2}$

ADDITIONAL EXAMPLES

Example 3

Write the expression $\left(16u^{\frac{2}{5}}v^{-4}\right)^{\frac{5}{4}}$ using only positive exponents. $\frac{32u^{\frac{1}{2}}}{v^5}$

Example 4

Simplify the expression $y^{\frac{1}{3}}\left(y^{\frac{1}{6}} - y^{\frac{5}{3}}\right)$. $y^{\frac{1}{2}} - y^2$

Example 5

Simplify the expression $\left(x^{\frac{7}{3}}y^3\right)\left(xy^{\frac{5}{6}}\right)^{-2}$. $x^{\frac{1}{3}}y^{\frac{4}{3}}$

Example 6

Let k be a positive rational number. Write the expression $\sqrt[12]{a^{4k}}\sqrt{\left(a^{-k}\right)^{\frac{1}{3}}}$ without radicals, using only positive exponents. $a^{\frac{k}{6}}$

Rationalizing Denominators and Numerators

Transforming fractions with radicals in the denominator to equivalent fractions with no radicals in the denominator is called *rationalizing the denominator*. Before the common use of calculators, fractions with rational denominators were preferred because they were easier to calculate or estimate. With calculators today there is no computational advantage to rationalizing denominators. However, the skill of rationalizing numerators or denominators is useful in calculus.

Example 7 Rationalizing the Denominator

Rationalize the denominator of each fraction.

a. $\frac{7}{\sqrt{5}}$ **b.** $\frac{2}{3 + \sqrt{6}}$

Solution

a. Multiply the fraction by 1 using a suitable radical fraction.

$$\frac{7}{\sqrt{5}} = \frac{7}{\sqrt{5}} \cdot 1 = \frac{7}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{7\sqrt{5}}{5}$$

b. Use the multiplication pattern $(a + b)(a - b) = a^2 - b^2$ to determine a suitable radical fraction equivalent to 1.

$$\frac{2}{3 + \sqrt{6}} = \frac{2}{3 + \sqrt{6}} \cdot 1$$
$$= \frac{2}{3 + \sqrt{6}} \cdot \frac{3 - \sqrt{6}}{3 - \sqrt{6}}$$
$$= \frac{2(3 - \sqrt{6})}{(3 + \sqrt{6})(3 - \sqrt{6})}$$
$$= \frac{6 - 2\sqrt{6}}{9 - 6}$$
$$= \frac{6 - 2\sqrt{6}}{3}$$

NOTE When rationalizing a denominator or numerator which contains a radical expression, use a suitable radical fraction, equal to one, that contains the conjugate of the expression.

Example 8 Rationalizing the Numerator

Assume $h \neq 0$. Rationalize the numerator of $\frac{\sqrt{x + h} - \sqrt{x}}{h}$.

Example Notes

The product in **Example 7b** has the same form as the product in Example 1d, page 329:

$(5 + \sqrt{c})(5 - \sqrt{c}) = 25 - c$

$(3 + \sqrt{6})(3 - \sqrt{6}) = 9 - 6$

COMMON ERROR ALERT

In **Example 7b,** students might attempt to mimic what was done in part **a.** That is, they might multiply the numerator and denominator by only $\sqrt{6}$. Remind students that $(3 + \sqrt{6})\sqrt{6}$ requires distributing $\sqrt{6}$, so $(3 + \sqrt{6})\sqrt{6} = 3\sqrt{6} + 6$. This does not eliminate the radical. Emphasize that they must use conjugates:

$(3 + \sqrt{6})(3 - \sqrt{6}) = 9 - 6 = 3$

Math Background

The expression in **Example 8** is an example of the difference quotient that students have encountered previously (pages 143, 220).

If $f(x) = \sqrt{x}$, then

$$\frac{f(x + h) - f(x)}{h} = \frac{\sqrt{x + h} - \sqrt{x}}{h}.$$

Students may have rationalized *denominators* in the past to transform fractions with radicals, but this is likely to be their first experience with rationalizing a *numerator*.

In calculus, students will learn the definition of the derivative of f at x:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

For $f(x) = \sqrt{x}$,

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h}$$

To find $f'(x)$, they will rationalize the numerator as shown in Example 8, and then evaluate the limit:

$$f'(x) = \lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}}$$
$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$
$$= \frac{1}{2\sqrt{x}}$$

Solution

Multiply the fraction by 1 using a suitable radical fraction.

$$\begin{aligned}\frac{\sqrt{x+h}-\sqrt{x}}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot 1\\ &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\ &= \frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\ &= \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\ &= \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}}\end{aligned}$$

Irrational Exponents

The example (not proof) below illustrates how a^t is defined when t is an irrational number.

To compute $10^{\sqrt{2}}$, the exponent could be replaced with the equivalent nonterminating decimal 1.414213562.... Each of the decimal approximations of $\sqrt{2}$ given below is a more accurate approximation than the preceding one.

$$1.4,\ 1.41,\ 1.414,\ 1.4142,\ 1.41421,\ \ldots.$$

We can raise 10 to each of these rational numbers.

$$\begin{aligned}10^{1.4} &\approx 25.1189\\ 10^{1.41} &\approx 25.7040\\ 10^{1.414} &\approx 25.9418\\ 10^{1.4142} &\approx 25.9537\\ 10^{1.41421} &\approx 25.9543\\ 10^{1.414213} &\approx 25.9545\end{aligned}$$

The pattern suggests that as the exponent r gets closer and closer to $\sqrt{2}$, 10^r gets closer and closer to a real number whose decimal expansion begins 25.954.... So $10^{\sqrt{2}}$ is defined to be this number.

Similarly, for any $a > 0$,

a^t is a well-defined positive number for each real exponent t.

The fact below shall be assumed.

The laws of exponents are valid for *all* real exponents.

ADDITIONAL EXAMPLES

Example 7

Rationalize the denominator of each fraction.

a. $\dfrac{5}{\sqrt{6}}$ $\dfrac{5\sqrt{6}}{6}$

b. $\dfrac{3}{4+\sqrt{5}}$ $\dfrac{12-3\sqrt{5}}{11}$

Example 8

Assume $h \neq 0$. Rationalize the numerator of $\dfrac{\sqrt{2t+h}-\sqrt{2t}}{h}$.

$\dfrac{1}{\sqrt{2t+h}+\sqrt{2t}}$

Teaching Notes

The laws of exponents on page 330 were stated for rational exponents. The section Irrational Exponents uses the concept of limit to show how to approximate $10^{\sqrt{2}}$, and then concludes by stating that the **laws of exponents are valid for *all* real exponents.**

Students might wonder why we do not simply approximate $10^{\sqrt{2}}$ directly with a calculator, since all the approximations listed require a calculator anyway! Point out that every power of 10 in the list of approximations has a rational exponent. For example, $10^{1.4} = 10^{\frac{14}{10}} = \left(\sqrt[10]{10}\right)^{14}$.

By observing that the successive approximations of $10^{\sqrt{2}}$ approach a limit, we can be sure that $10^{\sqrt{2}}$ has that limit as its unique (well-defined) value. Generalizing, we state that **a^t is a well-defined positive number for each real exponent t** and **the laws of exponents are valid for *all* real exponents.**

Irrational exponents will be used with exponential and logarithmic functions later in this chapter.