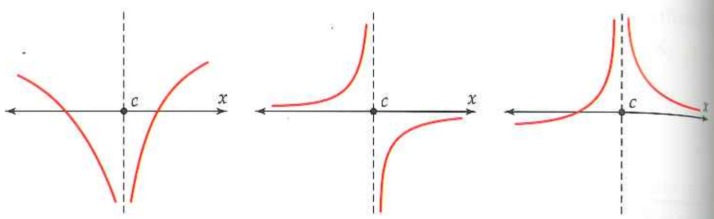
All rational functions have vertical asymptotes at values that are zeros of their denominators but not zeros of their numerators.

Vertical Asymptotes

A rational function has a vertical asymptote at $x = c$, provided

- **c is a zero of the denominator**
- **c is *not* a zero of the numerator**

Near a vertical asymptote, the graph of a rational fraction may look like the graph in Figure 4.4-2b, or like one of the graphs in Figure 4.4-3.



vertical asymptotes at $x = c$

Figure 4.4-3

Holes

When a number c is a zero of both the numerator and denominator of a rational function, the function might have a vertical asymptote at $x = c$, or it might behave differently.

You have often cancelled factors to reduce fractions.

$$\frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{x - 2} = x + 2$$

But the functions

$$p(x) = \frac{x^2 - 4}{x - 2} \qquad \text{and} \qquad q(x) = x + 2$$

are *not* the same, because when $x = 2$

$$p(2) = \frac{2^2 - 4}{2 - 2} = \frac{0}{0}, \text{ which is not defined, but}$$

$$q(x) = 2 + 2 = 4.$$

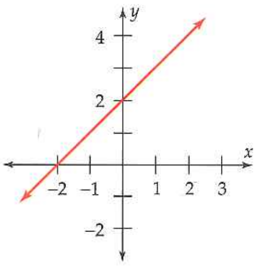


Figure 4.4-4a

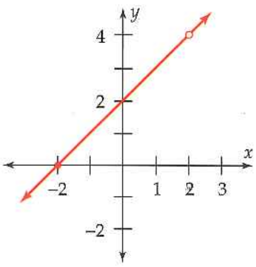
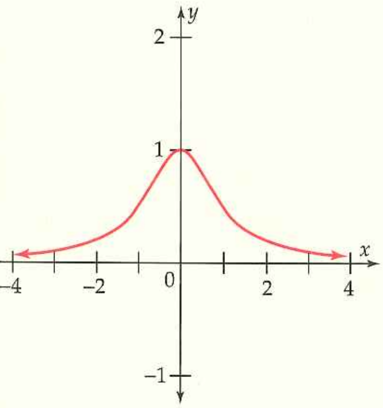


Figure 4.4-4b

For any number other than 2, the two functions have the same values, and hence, the same graphs. The graph of $q(x) = x + 2$ is a straight line that includes the point (2, 4), as shown in Figure 4.4-4a. The graph of $p(x)$ is the same straight line, but with the point (2, 4) omitted. That is, there is a **hole** in the graph of p at $x = 2$ because p is not defined there. The graph of p is shown in Figure 4.4-4b.

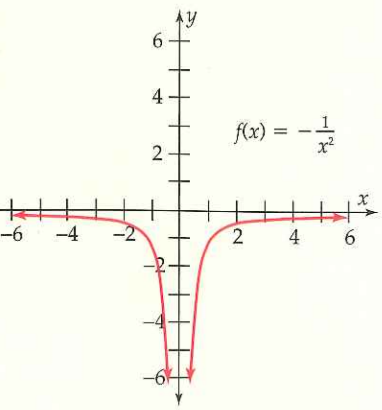
Math Background

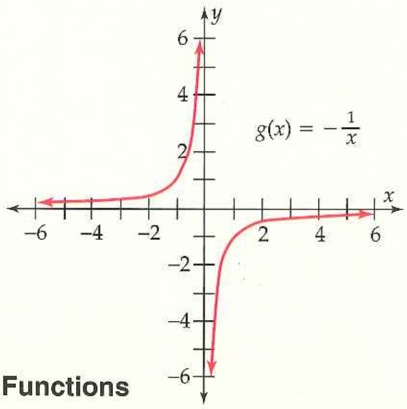
he first sentence on this page says, All rational functions have vertical symptotes at values that are zeros of eir denominators but not zeros of eir numerators." As with all other atements prior to this point, we are onsidering only the real number ystem. There *are* rational functions at have zeros of their denominators at are not zeros of their numerator nd *do not have* vertical asymptotes at ose values. However, such zeros are *naginary* zeros, which will be ddressed in Section 4.5 (Imaginary umbers are in the complex number ystem.). An example of such a nction is $f(x) = \frac{1}{x^2 + 1}$, shown elow. Its denominator has imaginary eros $\pm i$. Note that the graph has no ertical asymptotes.

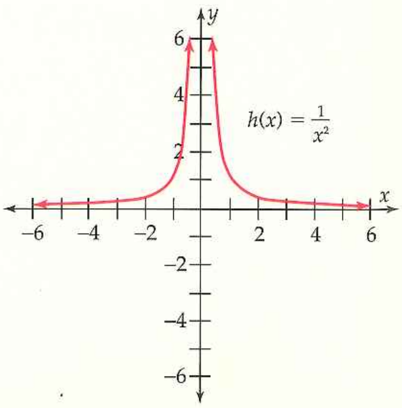


eaching Notes

ou may wish to show students xamples of functions that produce e types of vertical asymptotes own in Figure 4.4-3:







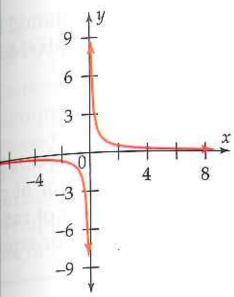


Figure 4.4-5

The graph of $g(x) = \frac{x^2}{x^3}$, shown in Figure 4.4-5, is the same as the graph of $f(x) = \frac{1}{x}$. At $x = 0$ neither function is defined. There is a vertical asymptote rather than a hole at $x = 0$. Note that the vertical asymptote occurs at $x = 0$, which is a zero of multiplicity 2 in the numerator, but of larger multiplicity 3 in the denominator.

Holes

Let $f(x) = \frac{g(x)}{h(x)}$ be a rational function and let d denote a zero of both g and h.

- **If the multiplicity of d as a zero of g is greater than or equal to its multiplicity as a zero of h, then the graph of f has a hole at $x = d$.**
- **Otherwise, the graph has a vertical asymptote at $x = d$.**

Accurate Rational Function Graphs

Getting an accurate graph of a rational function on a calculator often depends on choosing an appropriate viewing window. For example, the following are graphs of $f(x) = \frac{x+1}{2x-4}$ in different viewing windows.

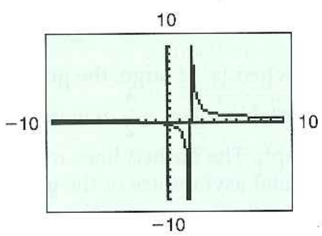


Figure 4.4-6a

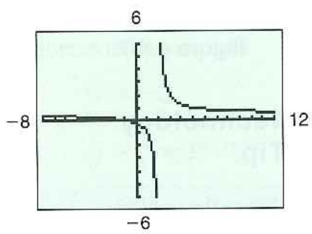


Figure 4.4-6b

Technology Tip

To avoid erroneous ver-
tical lines, use a
indow with a vertical
ymptote in the center
the screen. In Figure
-6b, the asymptote at
= 2 is halfway between
8 and 12. See the Tech-
logy Appendix for
rther information.

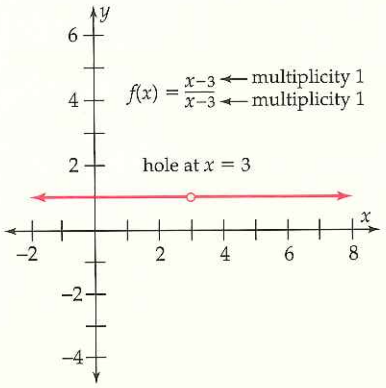
The vertical segment shown in Figure 4.4-6a is *not* a vertical asymptote. It is a result of the calculator evaluating f just to the left of $x = 2$ and just to the right of $x = 2$, but not at $x = 2$, and then erroneously connecting these points with a near vertical segment that looks like an asymptote. In the accurate graph shown in Figure 4.4-6b, the calculator attempted to plot a point with $x = 2$ and when it found that $f(2)$ was not defined, skipped a pixel and did not join the points on either side of the one skipped.

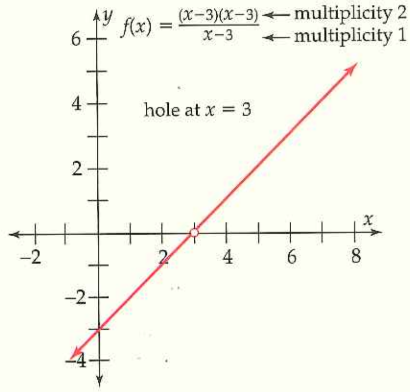
the Technology Tip, the asymptote $x = 2$ is
fway between -8 and 12
ause $\frac{-8+12}{2} = 2$.

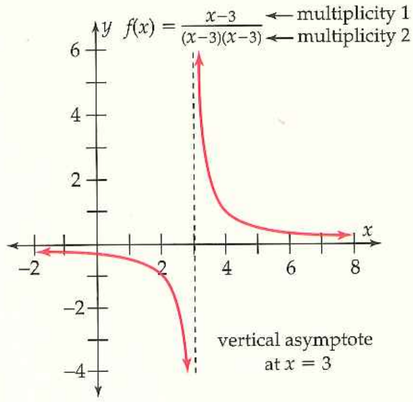
Teaching Notes

Review with students the definition of multiplicity: If $x - r$ is a factor that occurs m times in the complete factorization of a polynomial expression, then r is called a zero with multiplicity m of the related polynomial function (page 265).

You may want to provide students with examples of the different possibilities referred to in the **Holes** box:







Example Notes

In **Example 4a,** point out to students that the function value is never exactly $-\frac{3}{2}$.

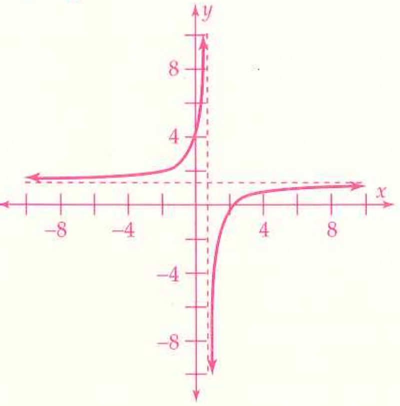
ADDITIONAL EXAMPLES

Example 4

List the vertical asymptotes and describe the end behavior of the following functions. Then sketch each graph.

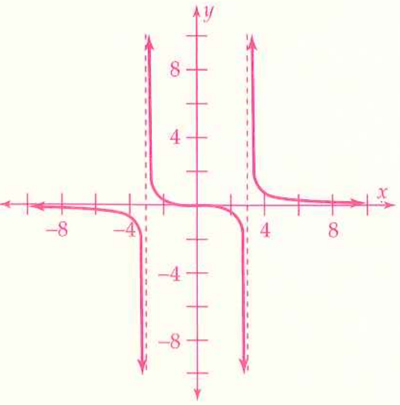
a. $f(x) = \dfrac{-4x + 8}{2 - 3x}$

vertical asymptote: $x = \frac{2}{3}$; When $|x|$ gets large, the graph of f gets very close to the horizontal asymptote $y = \frac{4}{3}$.



b. $g(x) = \dfrac{x}{x^2 - 9}$

vertical asymptotes: $x = 3$ and $x = -3$; When $|x|$ gets large, the graph of g gets very close to the horizontal asymptote $y = 0$.



c. $h(x) = \dfrac{3x^3 - x}{x^3 - 1}$

vertical asymptote: $x = 1$; When $|x|$ gets large, the graph of h gets very close to the horizontal asymptote $y = 3$.

A calculator graph may also fail to show holes in graphs that should have them. Even if a window is chosen so that the graph skips a pixel where the hole should be, the hole may be difficult to see.

End Behavior

As with polynomials, the behavior of a rational function when $|x|$ is large is called its **end behavior.** Known facts about the end behavior of polynomial functions make it easy to determine the end behavior of rational functions in which the degree of the numerator is less than or equal to the degree of the denominator.

Example 4 End Behavior of Rational Functions

List the vertical asymptotes and describe the end behavior of the following functions. Then sketch each graph.

a. $f(x) = \dfrac{3x - 6}{5 - 2x}$ **b.** $g(x) = \dfrac{x}{x^2 - 4}$ **c.** $h(x) = \dfrac{2x^3 - x}{x^3 + 1}$

Solution

a. The zero of the denominator of $f(x) = \dfrac{3x - 6}{5 - 2x}$ is $\frac{5}{2}$ and it is not a zero of the numerator. So the vertical asymptote occurs at $x = \frac{5}{2}$.

When $|x|$ is large, a polynomial function behaves like its highest degree term, as shown in Section 4.3. The highest degree term of the numerator of f is $3x$ and the highest degree term of the denominator is $-2x$. Therefore, when $|x|$ is large, the function reduces to approximately $-\frac{3}{2}$.

$$f(x) = \frac{3x - 6}{5 - 2x} = \frac{3x - 6}{-2x + 5} \approx \frac{3x}{-2x} = -\frac{3}{2}$$

Thus, when $|x|$ is large, the graph of f gets very close to the horizontal line $y = -\frac{3}{2}$, which is called a **horizontal asymptote** of the graph. The dashed lines in Figure 4.4-7 indicate the vertical and horizontal asymptotes of the graph.

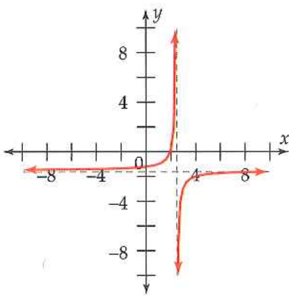


Figure 4.4-7

Technology Tip

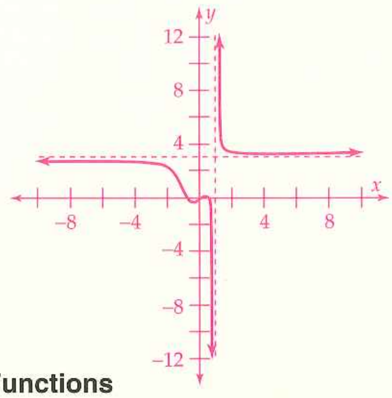
When the vertical asymptotes of a rational function occur at numbers such as $-2.1, -2, -1.9, \ldots, 2.9, 3$, etc., a decimal window normally produces an accurate graph because the calculator actually evaluates the function at the asymptotes, finds that it is undefined, and skips a pixel.

b. The zeros of the denominator of $g(x) = \dfrac{x}{x^2 - 4}$ are ± 2 and neither is a zero of the numerator. So the graph has vertical asymptotes at $x = -2$ and at $x = 2$.

When $|x|$ is large,

$$g(x) = \frac{x}{x^2 - 4} \approx \frac{x}{x^2} = \frac{1}{x}$$

and $\frac{1}{x}$ is very close to 0 by the Big-Little Concept. Therefore, the graph of g approaches the horizontal line $y = 0$ (the x-axis) when $|x|$ is large and this line is a horizontal asymptote of the graph, as shown in Figure 4.4-8.



CAUTION

Unlike a vertical asymptote that is never crossed by a graph, a graph may cross a horizontal or oblique asymptote at some values of $|x|$.

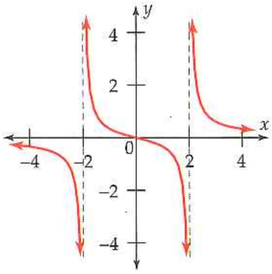


Figure 4.4-8

c. The only real zero of the denominator of $h(x) = \dfrac{2x^3 - x}{x^3 + 1}$ is $x = -1$, which is not a zero of the numerator. So, the graph has a vertical asymptote at $x = -1$.

When $|x|$ is large,

$$h(x) = \frac{2x^3 - x}{x^3 + 1} \approx \frac{2x^3}{x^3} = \frac{2}{1} = 2$$

Therefore, the graph of h has a horizontal asymptote at $y = 2$, as shown in Figure 4.4-9.

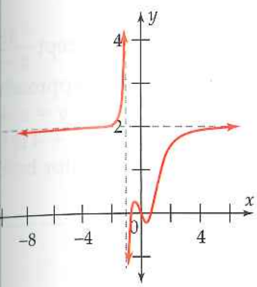


Figure 4.4-9

The function in Example **4b** illustrates a useful fact. When the degree of the numerator is less than the degree of the denominator of a rational function, the x-axis is the horizontal asymptote of the graph.

When the numerator and denominator have the same degree, as in Examples 4**a** and 4**c**, the horizontal asymptote is determined by the leading coefficients of the numerator and denominator:

Function	Horizontal asymptote
$f(x) = \dfrac{3x - 6}{-2x + 5}$	$y = -\dfrac{3}{2}$
$h(x) = \dfrac{2x^3 - x}{x^3 + 1}$	$y = \dfrac{2}{1} = 2$

Other Asymptotes

When the degree of the numerator of a rational function is greater than the degree of its denominator, the graph will not have a horizontal asymptote. To determine the end behavior in this case, the Division Algorithm must be used.

Example 5 **A Slant Asymptote**

Describe the end behavior of the graph of $f(x) = \dfrac{x^2 - x - 2}{x - 5}$.

5| 1 -1 -2
5 20
1 4 18
x + 4
(x - 5)(x + 4) 18

Teaching Notes

Tell students that the **Technology Tip** on page 284 may not apply if they use a built-in decimal window (or ZDecimal) feature for a function whose asymptote does not occur within the resulting window, such as $y = \dfrac{1}{x - 8}$. In such cases, the Technology Tip on page 283 would be better to use. For example, have them graph $y = \dfrac{1}{x - 8}$ using Xmin = 0 and Xmax = 16.

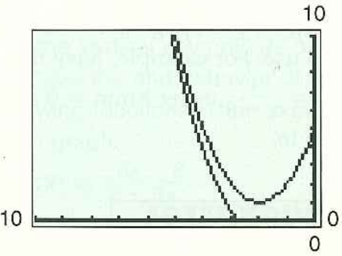
COMMON ERROR ALERT

Based on the way function f is written in **Example 4a,** some students may mistakenly identify the leading coefficients of the numerator and denominator as 3 and 5, respectively. They may say that a horizontal asymptote of $f(x) = \dfrac{3x - 6}{5 - 2x}$ is $y = \dfrac{3}{5}$. Remind students that the leading coefficient in a polynomial is the nonzero coefficient of the *highest power of the variable.* The correct horizontal asymptote is $y = -\dfrac{3}{2}$.

Some students may try to use the leading coefficients of the numerator and denominator to identify horizontal asymptotes when it is not an appropriate method. To illustrate, in **Example 4b,** some students may say that a horizontal asymptote of $g(x) = \dfrac{x}{x^2 - 4}$ is $y = \dfrac{1}{1} = 1$. Emphasize that when the degree of the numerator is less than the degree of the denominator, the x-axis, or $y = 0$, is the horizontal asymptote of the graph. (See **End Behavior of Rational Functions,** page 287.)

xample Notes

n **Example 6,** Figure 4.4-11, it appears
at the function graph and the
arabolic asymptotes coincide in some
laces. Students can try different
indow settings on a graphing
alculator, as shown below, to see
at the curves do not touch.



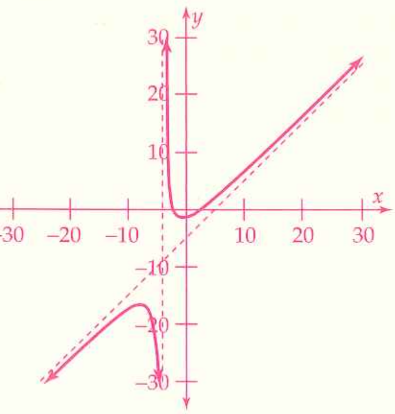
DDITIONAL EXAMPLES

xample 5

escribe the end behavior of the raph of $f(x) = \frac{x^2 - x - 6}{x + 4}$.

$x) = \frac{x^2 - x - 6}{x + 4} = (x - 5) + \frac{14}{x + 4}$

s $|x|$ gets large, the graph of f pproaches the slant asymptote $= x - 5$.



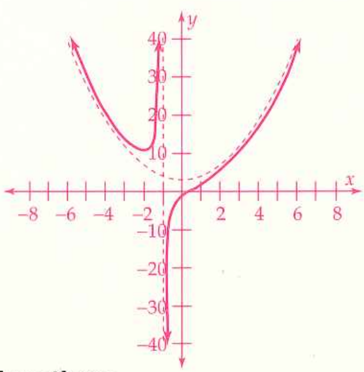
xample 6

escribe the end behavior of $x) = \frac{x^3 + x^2 + 3x - 1}{x + 1}$.

$x) = \frac{x^3 + x^2 + 3x - 1}{x + 1}$

$= (x^2 + 3) - \frac{4}{x + 1}$

s $|x|$ gets large, the graph of f pproaches the parabolic asymptote $= x^2 + 3$.



Solution

Use synthetic or long division to divide the denominator into the numerator, and rewrite the rational expression by using the Division Algorithm.

Dividend = Divisor Quotient + Remainder

$$x^2 - x - 2 = (x - 5)(x + 4) + 18$$

$$f(x) = \frac{x^2 - x - 2}{x - 5} = \frac{(x - 5)(x + 4) + 18}{x - 5} = \frac{(x - 5)(x + 4)}{x - 5} + \frac{18}{x - 5} = (x + 4) + \frac{18}{x - 5}$$

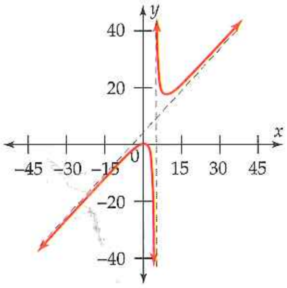


Figure 4.4-10

When $|x|$ is large, $x - 5$ is also large, and by the Big-Little Concept $\frac{18}{x - 5}$ is very close to 0. Therefore, $f(x) \approx x + 4$, and the graph of f approaches the line $y = x + 4$ as $|x|$ gets large (see Figure 4.4-10). The line $y = x + 4$ is called a **slant** or **oblique asymptote** of the graph. Note that $x + 4$ is the quotient without the remainder in the division of the numerator by the denominator.

Example 6 A Parabolic Asymptote

Describe the end behavior of the graph of $f(x) = \frac{x^3 + 3x^2 + x + 1}{x - 1}$.

Solution

Divide the denominator into the numerator and rewrite the function.

$$\textit{Quotient} + \frac{\textit{Remainder}}{\textit{Divisor}}$$

$$f(x) = \frac{x^3 + 3x^2 + x + 1}{x - 1} = (x^2 + 4x + 5) + \frac{6}{x - 1}$$

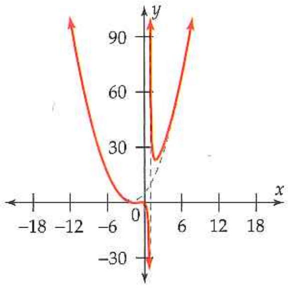


Figure 4.4-11

When $|x|$ is large, so is $x - 1$ and by the Big-Little Concept $\frac{6}{x - 1}$ is very close to 0. Therefore, $f(x) \approx x^2 + 4x + 5$ for large values of $|x|$. The graph of f approaches the parabola $y = x^2 + 4x + 5$, as shown in Figure 4.4-11. The curve $y = x^2 + 4x + 5$ is called a **parabolic asymptote.** Note that $x^2 + 4x + 5$ is the quotient in the division.

End Behavior of Rational Functions

Let $f(x) = \dfrac{ax^n + \cdots}{cx^k + \cdots}$ be a rational function whose numerator has degree n and whose denominator has degree k.

- **If $n < k$, then the x-axis is a horizontal asymptote.**
- **If $n = k$, then the line $y = \frac{a}{c}$ is a horizontal asymptote.**
- **If $n > k$, then the quotient polynomial when the numerator is divided by the denominator is the asymptote that describes the end behavior of the graph.**

Notice that when the degree of the numerator and the denominator are the same, the horizontal asymptote is the horizontal line determined by the quotient of the leading coefficients of the numerator and denominator.

Graphs of Rational Functions

The facts presented in this section can be used in conjunction with a calculator to find accurate, complete graphs of rational functions.

Graphing Rational Functions

1. **Analyze the function algebraically to determine its vertical asymptotes, holes, and intercepts.**
2. **Determine the end behavior of the graph.**

   If the degree of the numerator is less than or equal to the degree of the denominator, find the horizontal asymptote by using the facts in the box above.

   Otherwise, divide the numerator by the denominator. The quotient is the nonvertical asymptote of the graph.
3. **Use the preceding information to select an appropriate viewing window, or windows, to interpret the calculator's version of the graph, and display a complete graph of the function.**

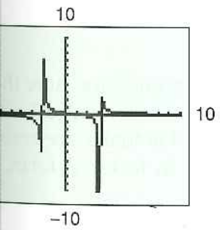


Figure 4.4-12a

Example 7 A Complete Graph of a Rational Function

Find a complete graph of $f(x) = \dfrac{x-1}{x^2 - x - 6}$.

Solution

The graph of f is shown in Figure 4.4-12a. It is hard to determine whether or not the graph is complete, so analyze the function algebraically.

Teaching Notes

The concepts in the **End Behavior of Rational Functions** box may seem intimidating, but they are really fairly simple. Write the following on the board:

$$f(x) = \frac{\Box x^{\Box} + \cdots}{\Box x^{\Box} + \cdots}$$

Now, fill in the boxes with numbers to illustrate each of the bulleted cases with examples such as:

- $f(x) = \dfrac{-7x^2 + \cdots}{2x^3 + \cdots}$

  (x-axis is a horizontal asymptote)
- $f(x) = \dfrac{3x^2 + \cdots}{2x^2 + \cdots}$

  ($y = \frac{3}{2}$ is a horizontal asymptote)
- $f(x) = \dfrac{x^2 - 21}{x + 5} = x - 5 + \dfrac{4}{x + 5}$

  ($y = x - 5$ is an asymptote)

You may want to review the following skills to prepare students for step 1 in the **Graphing Rational Functions** box:

- Factor the numerator and the denominator.
- After the numerator and denominator are factored, determine the domain of the function.
- Find the x- and y-intercepts.
- For excluded domain values, determine whether there are vertical asymptotes or holes.
- Find values of the function near any vertical asymptote, on both sides of the asymptote.

For step 2, determining end behavior, remind students of the 3 types of asymptotes:

- straight line horizontal
- straight line slant
- curved

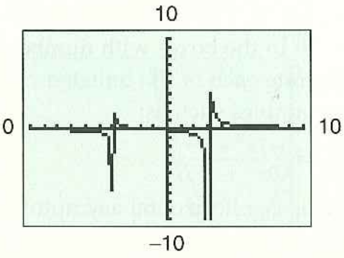
DDITIONAL EXAMPLES

xample 7

nd a complete graph of

$)=\frac{x+2}{x^2+x-12}$.

lculator graph:



alyze the function algebraically:

$)=\frac{x+2}{x^2+x-12}=\frac{x+2}{(x+4)(x-3)}$

ertical Asymptotes: $x=-4$ and $=3$.

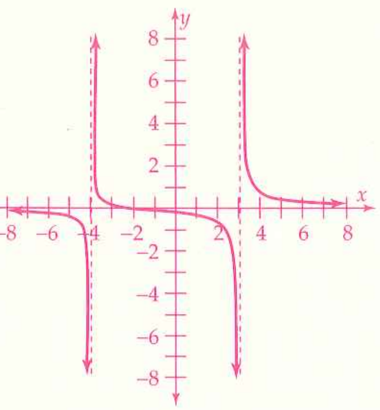
tercepts:

y-intercept: $f(0)=-\frac{1}{6}$

x-intercept: $x=-2$

orizontal Asymptote: $y=0$

mplete graph:



Begin by writing the function in factored form. Then read off the relevant information.

$$f(x)=\frac{x-1}{x^2-x-6}=\frac{x-1}{(x+2)(x-3)}$$

Vertical Asymptotes: $x=-2$ and $x=3$ — *zeros of the denominator but not the numerator*

Intercepts:

y-intercept: $f(0)=\frac{0-1}{0^2-0-6}=\frac{1}{6}$

x-intercept: $x=1$ — *zero of numerator but not of denominator*

Horizontal Asymptote: $y=0$ — *degree of numerator is less than degree of denominator*

Interpreting the above information suggests that a complete graph of *f* looks similar to Figure 4.4-12b.

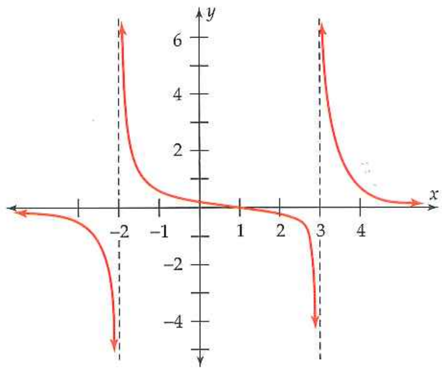


Figure 4.4-12b

Example 8 A Complete Graph of a Rational Function

Find a complete graph of $f(x)=\frac{x^3-2x^2-5x+6}{x^2+3x+2}$.

Solution

The denominator is easily factored. To factor the numerator, note that the only possible rational zeros of x^3-2x^2-5x+6 are ±1, ±2, ±3, and ±6 by the Rational Zeros Test. Verify that -2, 1 and 3 actually are zeros and use the Factor Theorem to write the numerator in factored form. Then reduce the fraction.

Holes:

$$f(x) = \frac{x^3 - 2x^2 - 5x + 6}{x^2 + 3x + 2} = \frac{(x + 2)(x - 1)(x - 3)}{(x + 2)(x + 1)}$$

$$= \frac{(x - 1)(x - 3)}{x + 1}, \text{ where } x \neq -2.$$

Therefore, the graph of f is the same as the graph of

$$g(x) = \frac{(x - 1)(x - 3)}{x + 1} = \frac{x^2 - 4x + 3}{x + 1}$$

except there is a hole when $x = -2$. Because

$$g(-2) = \frac{(-2 - 1)(-2 - 3)}{-2 + 1} = \frac{(-3)(-5)}{-1} = -15,$$

the hole occurs at $(-2, -15)$.

Intercepts:

y-intercept: $g(0) = \frac{(0 - 1)(0 - 3)}{0 + 1} = \frac{(-1)(-3)}{1} = 3$

x-intercepts: The x-intercepts of f are the same as the x-intercepts of g. Solving $(x - 1)(x - 3) = 0$ yields $x = 1$ or $x = 3$.

Vertical Asymptote: The vertical asymptote is $x = -1$.

End Behavior: Dividing the numerator by the denominator produces a quotient of $x - 5$. Therefore, the slant asymptote that describes the end behavior of the function is the line $y = x - 5$.

The graph of f is shown in Figure 4.4-13.

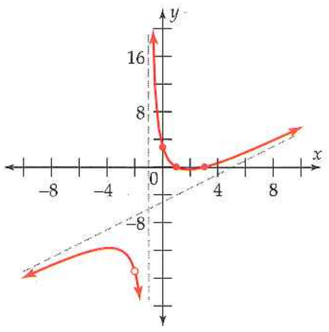


Figure 4.4-13

ADDITIONAL EXAMPLES

Example 8

Find a complete graph of $f(x) = \frac{x^3 - x^2 - 10x - 8}{x^2 - 7x + 12}$.

Possible rational zeros of $x^3 - x^2 - 10x - 8$:

$\pm 1, \pm 2, \pm 4, \pm 8$

Verify that the actual zeros are -1, -2 and 4.

$$f(x) = \frac{x^3 - x^2 - 10x - 8}{x^2 - 7x + 12}$$
$$= \frac{(x - 4)(x + 1)(x + 2)}{(x - 4)(x - 3)}$$
$$= \frac{(x + 1)(x + 2)}{(x - 3)}$$

Holes: The graph of f is the same as the graph of $g(x) = \frac{(x + 1)(x + 2)}{(x - 3)}$ except there is a hole when $x = 4$. Because $g(4) = 30$, the hole occurs at $(4, 30)$.

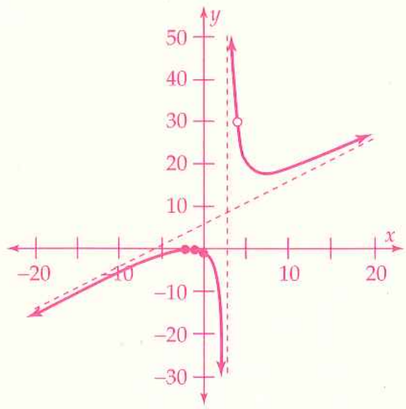
Intercepts:

y-intercept: $g(0) = -\frac{2}{3}$

x-intercepts: $x = -1$ and $x = -2$

Vertical Asymptote: $x = 3$

End Behavior: Dividing the numerator by the denominator produces a quotient of $x + 6$. Therefore, the slant asymptote that describes the end behavior of the function is the line $y = x + 6$.



ercises 4.4

NSWERS

All real numbers except $-\frac{5}{2}$

All real numbers except 3 and $-\frac{1}{2}$

All real numbers except $3 \pm \sqrt{5}$

All real numbers except 0 and $\pm\sqrt{6}$

All real numbers except $\pm\sqrt{2}$ and 1

All real numbers except $x \approx 1.00948$ and $x \approx -2.76232$

Vertical asymptotes $x = -1$ and $x = 6$

Vertical asymptotes $x = 0$ and $x = \frac{-7 \pm \sqrt{41}}{2}$

Hole at $x = 0$; vertical asymptote $x = -1$

Hole at $x = 0$; no vertical asymptotes

Vertical asymptote at $x = -2$; hole at $x = 2$

Hole at $x = 3$; vertical asymptote at $x = -2$

$y = 3$; any window with $-115 \le x \le 110$

$y = \frac{3}{2}$; any window with $-20 \le x \le 20$

$y = -1$; any window with $-31 \le x \le 35$

$y = 0$; any window with $-20 \le x \le 20$

$y = \frac{5}{2}$; any window with $-40 \le x \le 42$

$y = 16$; window: $-500 \le x \le 500$ and $0 \le y \le 20$

Asymptote: $y = x$; window: $-14 \le x \le 14$ and $-15 \le y \le 15$

$y = x^2 - 2x + 2$; any window with $-100 \le y \le 100$

Asymptote: $y = x^2 - x$; window: $-15 \le x \le 6$ and $-40 \le y \le 240$

$y = x + 4$; standard viewing window

24 – 34 evens 27, 33, 40, 44, 46 (29)

Exercises 4.4

In Exercises 1–6, find the domain of the function.

1. $f(x) = \dfrac{-3x}{2x + 5}$ **2.** $g(x) = \dfrac{x^3 + x + 1}{2x^2 - 5x - 3}$

3. $h(x) = \dfrac{6x - 5}{x^2 - 6x + 4}$ **4.** $g(x) = \dfrac{x^3 - x^2 - x - 1}{x^5 - 36x}$

5. $f(x) = \dfrac{x^5 - 2x^3 + 7}{x^3 - x^2 - 2x + 2}$

6. $h(x) = \dfrac{x^5 - 5}{x^4 + 12x^3 + 60x^2 + 50x - 125}$

In Exercises 7–12, use algebra to determine the location of the vertical asymptotes and holes in the graph of the function.

7. $f(x) = \dfrac{x^2 + 4}{x^2 - 5x - 6}$ **8.** $g(x) = \dfrac{x - 5}{x^3 + 7x^2 + 2x}$

9. $f(x) = \dfrac{x}{x^3 + 2x^2 + x}$ **10.** $g(x) = \dfrac{x}{x^3 + 5x}$

11. $f(x) = \dfrac{x^2 - 4x + 4}{(x + 2)(x - 2)^3}$ **12.** $h(x) = \dfrac{x - 3}{x^2 - x - 6}$

In Exercises 13–22, find the horizontal or other asymptote of the graph of the function when $|x|$ is large, and find a viewing window in which the ends of the graph are within 0.1 of this asymptote.

13. $f(x) = \dfrac{3x - 2}{x + 3}$ **14.** $g(x) = \dfrac{3x^2 + x}{2x^2 - 2x + 4}$

15. $h(x) = \dfrac{5 - x}{x - 2}$ **16.** $f(x) = \dfrac{4x^2 - 5}{2x^3 - 3x^2 + x}$

17. $g(x) = \dfrac{5x^3 - 8x^2 + 4}{2x^3 + 2x}$

18. $h(x) = \dfrac{8x^5 - 6x^3 + 2x - 1}{0.5x^5 + x^4 + 3x^2 + x}$ **19.** $f(x) = \dfrac{x^3 - 1}{x^2 - 4}$

20. $g(x) = \dfrac{x^3 - 4x^2 + 6x + 5}{x - 2}$

21. $h(x) = \dfrac{x^3 + 3x^2 - 4x + 1}{x + 4}$

22. $f(x) = \dfrac{x^3 + 3x^2 - 4x + 1}{x^2 - x}$

In Exercises 23–50, analyze the function algebraically, list its vertical asymptotes, holes, and horizontal asymptote. Then sketch a complete graph of the function.

23. $f(x) = \dfrac{1}{x + 5}$ **24.** $q(x) = \dfrac{-7}{x - 6}$

25. $k(x) = \dfrac{-3}{2x + 5}$ **26.** $g(x) = \dfrac{-4}{2 - x}$

27. $f(x) = \dfrac{3x}{x - 1}$ **28.** $p(x) = \dfrac{x - 2}{x}$

29. $f(x) = \dfrac{2 - x}{x - 3}$ **30.** $g(x) = \dfrac{3x - 2}{x + 3}$

31. $f(x) = \dfrac{1}{x(x + 1)^2}$ **32.** $g(x) = \dfrac{x}{2x^2 - 5x - 3}$

33. $f(x) = \dfrac{x - 3}{x^2 + x - 2}$ **34.** $g(x) = \dfrac{x + 2}{x^2 - 1}$

35. $h(x) = \dfrac{(x^2 + 6x + 5)(x + 5)}{(x + 5)^3(x - 1)}$

36. $f(x) = \dfrac{x^2 - 1}{x^3 - 2x^2 + x}$ **37.** $f(x) = \dfrac{-4x^2 + 1}{x^2}$

38. $k(x) = \dfrac{x^2 + 1}{x^2 - 1}$ **39.** $q(x) = \dfrac{x^2 + 2x}{x^2 - 4x - 5}$

40. $F(x) = \dfrac{x^2 + x}{x^2 - 2x + 4}$

41. $p(x) = \dfrac{(x + 3)(x - 3)}{(x - 5)(x + 4)(x + 3)}$

42. $p(x) = \dfrac{x^3 + 3x^2}{x^4 - 4x^2}$ **43.** $f(x) = \dfrac{x^2 - x - 6}{x - 2}$ Ex

44. $k(x) = \dfrac{x^2 + x - 2}{x}$ **45.** $Q(x) = \dfrac{4x^2 + 4x - 3}{2x - 5}$

46. $K(x) = \dfrac{3x^2 - 12x + 15}{3x + 6}$ **47.** $f(x) = \dfrac{x^3 - 2}{x - 1}$

48. $p(x) = \dfrac{x^3 + 8}{x + 1}$ **49.** $q(x) = \dfrac{x^3 - 1}{x - 2}$

50. $f(x) = \dfrac{x^4 - 1}{x^2}$

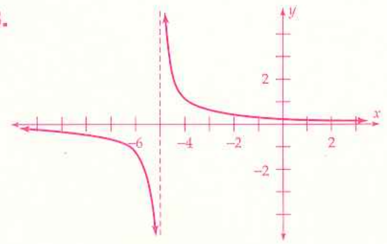
In Exer… that sh… asympt… alert fo…

51. $f(x$ … 52. $g($ … 54. $f($ … 55. $f($ … 57. $h($ … 59. $g($ … 60. $h($ …

In Exe… that sh… sible, … alert f…

61. $f($ … 63. $h($ … 64. f … 65. g … 66. h … 67. a …

23.



vertical asymptote $x = -5$

horizontal asymptote $y = 0$

24–50. *See pp.* **1063–1066.**

Exercises 51–60, find a viewing window or windows at show(s) a complete graph of the function using ymptotes, intercepts, end behavior, and holes. Be rt for hidden behavior.

. $f(x) = \dfrac{x^3 + 4x^2 - 5x}{(x^2 - 4)(x^2 - 9)}$

. $g(x) = \dfrac{x^2 + x - 6}{x^3 - 19x + 30}$ **53.** $h(x) = \dfrac{2x^2 - x - 6}{x^3 + x^2 - 6x}$

. $f(x) = \dfrac{x^3 - x + 1}{x^4 - 2x^3 - 2x^2 + x - 1}$

. $f(x) = \dfrac{2x^4 - 3x^2 + 1}{3x^4 - x^2 + x - 1}$ **56.** $g(x) = \dfrac{x^4 + 2x^3}{x^5 - 25x^3}$

. $h(x) = \dfrac{3x^2 + x - 4}{2x^2 - 5x}$ **58.** $f(x) = \dfrac{2x^2 - 1}{3x^3 + 2x + 1}$

. $g(x) = \dfrac{x - 4}{2x^3 - 5x^2 - 4x + 12}$

. $h(x) = \dfrac{x^2 - 9}{x^3 + 2x^2 - 23x - 60}$

Exercises 61–66, find a viewing window or windows at show(s) a complete graph of the function—if posle, with no erroneous vertical line segments. Be rt for hidden behavior.

. $f(x) = \dfrac{2x^2 + 5x + 2}{2x + 7}$ **62.** $g(x) = \dfrac{2x^3 + 1}{x^2 - 1}$

$h(x) = \dfrac{x^3 - 2x^2 + x - 2}{x^2 - 1}$

$f(x) = \dfrac{3x^3 - 11x - 1}{x^2 - 4}$

$g(x) = \dfrac{2x^4 + 7x^3 + 7x^2 + 2x}{x^3 - x + 50}$

$h(x) = \dfrac{2x^3 + 7x^2 - 4}{x^2 + 2x - 3}$

a. Graph $f(x) = \dfrac{1}{x}$ in the viewing window with $-6 \le x \le 6$ and $-6 \le y \le 6$.

b. Without using a calculator, describe how the graph of $g(x) = \dfrac{2}{x}$ can be obtained from the graph of $f(x)$. *Hint:* $g(x) = 2f(x)$

c. Without using a calculator, describe how the graphs of each of the following functions can be obtained from the graph of $f(x)$.

$h(x) = \dfrac{1}{x} + 4 \qquad k(x) = \dfrac{1}{x - 3} \qquad t(x) = \dfrac{1}{x + 2}$

d. Without using a calculator, describe how the graph of $p(x) = \dfrac{2}{x - 3} + 4$ can be obtained from the graph of $f(x) = \dfrac{1}{x}$.

e. Show that the function $p(x)$ of part **d** is a rational function by rewriting its rule as the quotient of two first-degree polynomials.

f. If r, s, and t are constants, describe how the graph of $q(x) = \dfrac{r}{x + s} + t$ can be obtained from the graph of $f(x) = \dfrac{1}{x}$.

g. Show that the function $q(x)$ of part f is a rational function by rewriting its rule as the quotient of two first-degree polynomials.

68. The graph of $f(x) = \dfrac{2x^3 - 2x^2 - x + 1}{3x^3 - 3x^2 + 2x - 1}$ has a vertical asymptote. Find a viewing window that demonstrates this fact.

69. a. Find the difference quotient of $f(x) = \dfrac{1}{x}$ and express it as a single fraction in lowest terms.

b. Use the difference quotient in part **a** to determine the average rate of change of $f(x)$ as x changes from 2 to 2.1, from 2 to 2.01, and from 2 to 2.001. Estimate the instantaneous rate of change of $f(x)$ at $x = 2$.

c. Use the different quotient in part **a** to determine the average rate of change of $f(x)$ as x changes from 3 to 3.1, from 3 to 3.01, and from 3 to 3.001. Estimate the instantaneous rate of change of $f(x)$ at $x = 3$.

d. How are the estimated instantaneous rates of change of $f(x)$ at $x = 2$ and $x = 3$ related to the values of $g(x) = \dfrac{-1}{x^2}$ at $x = 2$ and $x = 3$?

70. Do Exercise 69 for the functions $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{-2}{x^3}$.

71. a. When $x \ge 0$, what rational function has the same graph as $f(x) = \dfrac{x - 1}{|x| - 2}$? *Hint:* Use the definition of absolute value.

b. When $x < 0$, what rational function has the same graph as $f(x) = \dfrac{x - 1}{|x| - 2}$? See the hint for part **a**.

51. Overall: $-5 \le x \le 4.4$ and $-8 \le y \le 4$; hidden area near origin: $-2 \le x \le 2$ and $-0.5 \le y \le 0.5$; hidden area near $x = -5$: $-15 \le x \le -3$ and $-0.07 \le y \le 0.02$

52. $-8 \le x \le 8$ and $-5 \le y \le 5$

53. $-9.4 \le x \le 9.4$ and $-4 \le y \le 4$; there is a hole at $x = 2$.

54. $-5 \le x \le 8$ and $-3 \le y \le 2$

55. Overall: $-4.7 \le x \le 4.7$ and $-2 \le y \le 2$; there is a hole at $x = -1$; to see the vertical asymptote, use $0.65 \le x \le 0.75$ and $-3 \le y \le 3$.

56. $-12 \le x \le 12$ and $-4 \le y \le 4$ there is a hole at $x = 0$

57. For vertical asymptotes and x-intercepts: $-4.7 \le x \le 4.7$ and $-8 \le y \le 8$; to see graph get close to the horizontal asymptote: $-40 \le x \le 35$ and $-2 \le y \le 3$

58. $-5 \le x \le 5$ and $-5 \le y \le 5$

59. Overall: $-4.7 \le x \le 4.7$ and $-2 \le y \le 2$; hidden area near $x = 4$: $3 \le x \le 15$ and $-0.02 \le y \le 0.01$

60. $-8 \le x \le 8$ and $-8 \le y \le 8$; there is a hole at $x = -3$

61. $-15.5 \le x \le 8.5$ and $-16 \le y \le 8$

62. $-5 \le x \le 5$ and $-7 \le y \le 7$

63. $-4.7 \le x \le 4.7$ and $-12 \le y \le 8$

64. $-4 \le x \le 4$ and $-12 \le y \le 12$

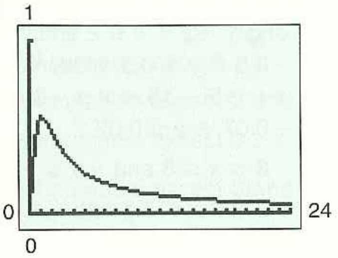
65. Overall: $-13 \le x \le 7$ and $-20 \le y \le 20$; hidden area near the origin: $-2.5 \le x \le 1$ and $-0.02 \le y \le 0.02$

66. $-6 \le x \le 6$ and $-12 \le y \le 12$

67–71. *See p. 1066.*

a. about 26%

b.



c. The horizontal asymptote is $c = 0$. The amount of the drug in the bloodstream gets vanishingly small.

d. There is about 55.9% in the blood at time 1.12.

a. $C(x) = \frac{3}{100}(2x^2) + \frac{1.25}{100}\left(4x \cdot \frac{1000}{x^2}\right) = 0.06x^2 + \frac{50}{x}$

b. $x \approx 16.85$ in.

a. $C(x) = \frac{2800 + 3.5x^2}{x}$

b. speeds of about 13.91 and 57.52 miles per hour

c. about 28.28 miles per hour

a. $c(x) = \frac{20 + x}{50 + x}$

b. between 25 gallons and 100 gallons

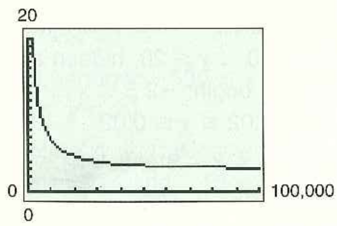
c. $x = 50$ gallons

a. $P(x) = \frac{500 + x^2}{x}$

b. between 10 and 50 meters

a. $a(x) = \frac{c(x)}{x} = \frac{40{,}000 + 2.60x}{x}$

b.



c. $y = 2.60$; the average cost can never be below \$2.60.

a. $h_1 = h - 2$

b. $h_1 = \frac{150}{\pi r^2} - 2$

c. $V = \pi(r - 1)^2\left(\frac{150}{\pi r^2} - 2\right)$

d. r must be more than 1 since the walls are 1 foot thick.

---

c. Use parts **a** and **b** to explain why the graph of $f(x) = \frac{x - 1}{|x| - 2}$ has two vertical asymptotes. What are they? Confirm your answer by graphing the function.

72. The percentage c of a drug in a person's bloodstream t hours after its injection is approximated by $c(t) = \frac{5t}{4t^2 + 5}$.

a. Approximately what percentage of the drug is in the person's bloodstream after four and a half hours?

b. Graph the function c in an appropriate window for this situation.

c. What is the horizontal asymptote of the graph? What does it tell you about the amount of the drug in the bloodstream?

d. At what time is the percentage the highest? What is the percentage at that time?

73. A box with a square base and a volume of 1000 cubic inches is to be constructed. The material for the top and bottom of the box costs \$3 per 100 square inches and the material for the sides costs \$1.25 per 100 square inches.

a. If x is the length of a side of the base, express the cost of constructing the box as a function of x.

b. If the side of the base must be at least 6 inches long, for what value of x will the cost of the box be \$20?

74. A truck traveling at a constant speed on a reasonably straight, level road burns fuel at the rate of $g(x)$ gallons per mile, where x is the speed of the truck in miles per hour and $g(x)$ is given by $g(x) = \frac{800 + x^2}{200x}$.

a. If fuel costs \$1.40 per gallon, find the rule of the cost function $c(x)$ that expresses the cost of fuel for a 500-mile trip as a function of the speed. *Hint:* $500 \cdot g(x)$ gallons of fuel are needed to go 500 miles. (Why?)

b. What driving speed will make the cost of fuel for the trip \$250?

c. What driving speed will minimize the cost of fuel for the trip?

75. Pure alcohol is being added to 50 gallons of a coolant mixture that is 40% alcohol.

a. Find the rule of the concentration function $c(x)$ that expresses the percentage of alcohol in the resulting mixture as a function of the number x of gallons of pure alcohol that are added. *Hint:* The final mixture contains $50 + x$ gallons. (Why?) So $c(x)$ is the amount of alcohol in the final mixture divided by the total amount $50 + x$. How much alcohol is in the original 50-gallon mixture? How much is in the final mixture?

b. How many gallons of pure alcohol should be added to produce a mixture that is at least 60% alcohol and no more than 80% alcohol?

c. Determine algebraically the exact amount of pure alcohol that must be added to produce a mixture that is 70% alcohol.

76. A rectangular garden with an area of 250 square meters is to be located next to a building and fenced on three sides, with the building acting as a fence on the fourth side.

a. If the side of the garden parallel to the building has length x meters, express the amount of fencing needed as a function of x.

b. For what values of x will less than 60 meters of fencing be needed?

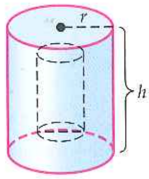
77. A certain company has fixed costs of \$40,000 and variable costs of \$2.60 per unit.

a. Let x be the number of units produced. Find the rule of the average cost function. (The average cost is the cost of the units divided by the number of units.)

b. Graph the average cost function in a window with $0 \le x \le 100{,}000$ and $0 \le y \le 20$.

c. Find the horizontal asymptote of the average cost function. Explain what the asymptote means in this situation, that is, how low can the average cost possibly be?

78. Radioactive waste is stored in a cylindrical tank, whose exterior has radius r and height h as shown in the figure. The sides, top, and bottom of the tank are one foot thick and the tank has a volume of 150 cubic feet including top, bottom, and walls.



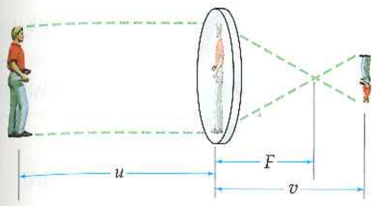
a. Express the interior height h_1 (that is, the height of the storage area) as a function of h.

b. Express the interior height as a function of r.

c. Express the volume of the interior as a function of r.

d. Explain why r must be greater than 1.

The relationship between the fixed focal length F of a camera, the distance u from the object being photographed to the lens, and the distance v from the lens to the film is given by $\frac{1}{F} = \frac{1}{u} + \frac{1}{v}$.



a. If the focal length is 50 mm, express v as a function of u.

b. What is the horizontal asymptote of the graph of the function in part **a**?

c. Graph the function in part **a** when $50 \text{ mm} < u < 35{,}000 \text{ mm}$.

d. When you focus the camera on an object, the distance between the lens and the film is changed. If the distance from the lens to the camera changes by less than 0.1 millimeter, the object will remain in focus. Explain why you have more latitude in focusing on distant objects than on very close ones.

80. The formula for the gravitational acceleration in units of meters per second squared of an object relative to the earth is

$$g(r) = \frac{3.987 \times 10^{14}}{(6.378 \times 10^6 + r)^2}$$

where r is the distance in meters above the earth's surface.

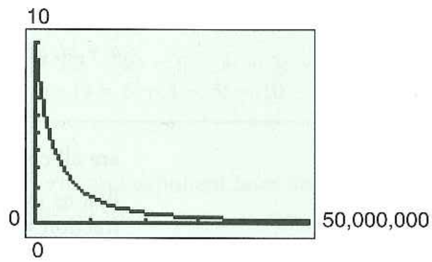
a. What is the gravitational acceleration at the earth's surface?

b. Graph the function $g(r)$ for $r \geq 0$.

c. Can you ever escape the pull of gravity? Does the graph have any r-intercepts?

a. **≈9.801 meters per second per second**

b.



c. no; no

.5 Complex Numbers

ectives

- /rite complex numbers in :andard form
- erform arithmetic perations on complex umbers
- ind the conjugate of a omplex number
- implify square roots of egative numbers
- nd all solutions of olynomial equations

If restricted to nonnegative numbers, you cannot solve the equation $x + 5 = 0$. Enlarging the number system to include negative integers makes it possible to find the solution to this equation. By enlarging the number system to include rational numbers, it is possible to solve equations that have no integer solution, such as $3x = 7$. Similarly, the equation $x^2 = 2$ has no rational solution, but $x = \sqrt{2}$ and $x = -\sqrt{2}$ are real number solutions. The idea of enlarging a number system to include solutions to equations that cannot be solved in a particular number system is a natural one.

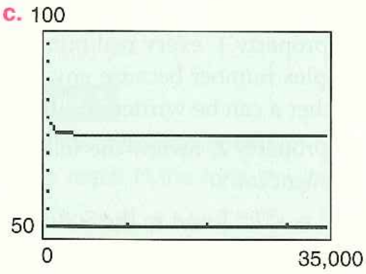
Complex Numbers

Equations such as $x^2 = -1$ and $x^2 = -4$ have no solutions in the real number system because $\sqrt{-1}$ and $\sqrt{-4}$ are not real numbers. In order to solve such equations, that is, to find the square roots of negative numbers, the number system must be enlarged again. There is a number system, called the **complex number system,** with the desired properties.

79. a. $v = \frac{50u}{u - 50}$

b. $v = 50$

c.



d. If the object is close, a small change in u leads to a large change in v. However, when u is large, a small change in u leads to nearly no change in v, so that u may change substantially while the object stays in focus.

Section 4.5 Complex Numbers

Teaching Notes

Point out that when you build a new number system from a smaller one, the new one should contain the smaller one. In particular, all natural numbers are integers, all integers are rational numbers, all rational numbers are real numbers, and, in the present setting, all real numbers are complex numbers.

You may want to review (from Section 2.2):

- The solutions of a quadratic equation $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- If $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is not a real number, and the equation has no real solutions.

In this section, students will learn that when $b^2 - 4ac < 0$, there are solutions of $ax^2 + bx + c = 0$ that are complex numbers.