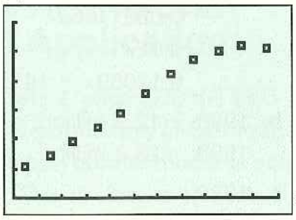
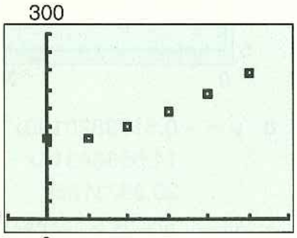
11. a.



b. $y = -0.084189248x^4 - 0.528069153x^3 + 66.26642628x^2 + 397.2751554x + 3965.686061$

c. 1996: $19,606; The estimate is lower.

12. a.



b. $y = 0.83804x^2 + 2.53493x + 127.77571$

c. 1995: $161.40; 2002: $278.87

d. during the year 2004

Section 4.4 Rational Functions

Teaching Notes

For this lesson, students need to know how to factor and divide polynomials and how to find zeros and intercepts of polynomial functions.

Emphasize that the domains of most rational functions exclude one or more real numbers.

d. Use the model to predict the median income in 2002.

e. Does this model seem reasonable after 2002?

10. a. Sketch a scatter plot of the data from 1989 to 1999, with $x = 0$ corresponding to 1989.

b. Find both a cubic and a quartic model for this data.

c. Is there any significant difference between the models from 1989 to 1999? What about from 1999 to 2005?

d. According to these models, when will the median income reach $45,000?

11. The table shows the U.S. public debt per person, in dollars, in selected years.

Year	Debt	Year	Debt
1981	$4,338	1993	$17,105
1983	5,870	1995	18,930
1985	7,598	1997	20,026
1987	9,615	1999	20,746
1989	11,545	2001	20,353
1991	14,436		

[*Source: U.S. Department of Treasury, Bureau of Public Debt*]

a. Sketch a scatter plot of the data with $x = 0$ corresponding to 1980.

b. Find a quartic model for the data.

c. Use the model to estimate the public debt per person in 1996. How does your estimate compare with the actual figure of $19,805?

12. The table shows the total advertising expenditures, in billions of dollars, in selected years.

Year	Expenditures	Year	Expenditures
1990	$129.59	1996	$175.23
1992	132.65	1998	201.59
1994	151.68	2000	236.33

[*Source: Statistical abstract of the United States 2001*]

a. Sketch a scatter plot of the data with $x = 0$ corresponding to 1990.

b. Find a quadratic model for the data.

c. Use the model to estimate expenditures in 1995 and 2002.

d. If this model remains accurate, when will expenditures reach $350 billion?

4.4 Rational Functions

Objectives

- Find the domain of a rational function
- Find intercepts, vertical asymptotes, and horizontal asymptotes
- Identify holes
- Describe end behavior
- Sketch complete graphs

Recall that a polynomial is an algebraic expression that can be written as

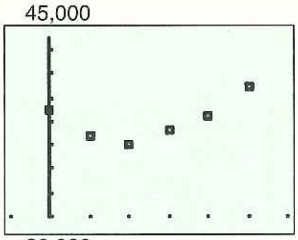
$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

where n is a nonnegative integer.

A **rational function** is a function whose rule is the quotient of two polynomials, such as

$$f(x) = \frac{1}{x} \qquad t(x) = \frac{4x - 3}{2x + 1} \qquad k(x) = \frac{2x^3 + 5x + 2}{x^2 - 7x + 6}$$

10. a.



b. Cubic: $y = -6.8924x^3 + 236.9524x^2 - 1487.44x + 38{,}847.167$

Quartic: $y = 0.5326x^4 - 17.543x^3 + 302.38x^2 - 1609.1687x + 38{,}861.77$

c. There is not a significant difference between the models from 1989 to 1999; between 1999 and 2005, the quartic model increases faster.

d. Cubic: in 2002; quartic: in 2001

Although a polynomial function is defined for every real number x, a rational function is defined only when its denominator is nonzero.

Domain of Rational Functions

> **The domain of a rational function is the set of all real numbers that are *not* zeros of its denominator.**

Example 1 **The Domain of a Rational Function**

Find the domain of each rational function.

a. $f(x) = \frac{1}{x^2}$ **b.** $g(x) = \frac{x^2 + 3x + 1}{x^2 - x - 6}$

Solution

a. The domain of $f(x) = \frac{1}{x^2}$ is the set of all real numbers except $x = 0$, because the denominator is 0 when $x = 0$, making the fraction undefined.

b. The domain of $g(x) = \frac{x^2 + 3x + 1}{x^2 - x - 6}$ is the set of all real numbers except the solutions of $x^2 - x - 6 = 0$. Because $x^2 - x - 6$ factors into $(x + 2)(x - 3)$, the solutions to $x^2 - x - 6 = 0$ are $x = -2$ and $x = 3$. Therefore, the domain of g is the set of all real numbers except $x = -2$ and $x = 3$.

Properties of Rational Graphs

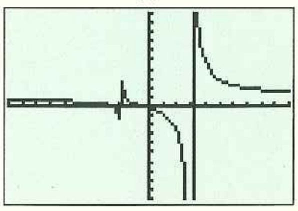
Because calculators often do a poor job of graphing rational functions, the emphasis in this section is on the algebraic analysis of rational functions. Such analysis should enable you to interpret misleading screen images.

Intercepts

As with any function, the y-intercept of the graph of a rational function f occurs at $f(0)$, provided that f is defined at $x = 0$. The x-intercepts of the graph of a rational function occur when its numerator is 0 and its denominator is nonzero.

Intercepts of Rational Functions

> **If f has a y-intercept, it occurs at $f(0)$.**
>
> **The x-intercepts of the graph of a rational function occur at the numbers that**
>
> - **are zeros of the numerator**
> - **are *not* zeros of the denominator**



COMMON ERROR ALERT

Some students may exclude zeros of the numerator from the domain of a rational function. For example, they may exclude 3 from the domain of $t(x) = \frac{2x - 6}{2x + 1}$. Remind students that the domain of a rational function is the set of all real numbers that are not zeros of its *denominator.* The domain of t is the set of all real numbers except $x = -\frac{1}{2}$.

Math Background

In **Example 1a,** the domain is described as the set of all real numbers except $x = 0$. Other notations that describe this set include:

- $\{x | x \neq 0\}$ (set notation)
- $(-\infty, 0) \cup (0, \infty)$ (interval notation)

ADDITIONAL EXAMPLES

Example 1

Find the domain of each rational function.

a. $f(x) = \frac{1}{x^3}$ all real numbers except $x = 0$

b. $\frac{x^2 + 4}{x^2 - 2x - 8}$ all real numbers except $x = -2$ and $x = 4$

Teaching Notes

Have students use their calculators to graph the function in **Example 1b,** and then direct their attention to the paragraph under **Properties of Rational Graphs.** The calculator probably does a poor job with the graph of the function g. Point out that if their graph of g appears as shown to the left, the vertical segments are not really part of the graph. (Possible ways to avoid these "extra" line segments are given in the **Technology Tips** on pages 283 and 284.)

Teaching Notes

After doing Example 2, you may want to point out that not all rational functions have x- and/or y-intercepts. For example, the function $f(x) = \frac{1}{x}$ has neither an x-intercept nor a y-intercept.

The paragraph under **Continuity** mentions breaks in graphs of rational functions. There are two types of breaks. One type of break occurs at a point where a **vertical asymptote** may be drawn. Vertical asymptotes are defined at the bottom of page 281. The other type of break occurs at a point that is a hole in the function (see page 283).

After learning about vertical asymptotes, some students may ask about other types of asymptotes. For horizontal asymptotes; see page 284; for slant/oblique and parabolic asymptotes, see pages 285–286. You may want to mention that the restriction that the graph never crosses an asymptote applies only to vertical asymptotes.

Example Notes

In Example 3, Figure 4.4-2a, the values for X in the table must be entered manually into the calculator.

Math Background

Under **Continuity**, note that except for breaks where the function is not defined, a rational function graph is a continuous unbroken curve. Students may ask about the graph in Figure 4.4-4b, which is a line with a hole in it. The broad meaning of *curve* in mathematics is a one-dimensional figure obtained by transforming a line or line segment. Applying this broad meaning, a line or line segment itself is a special case of a curve.

The method used in Example 3—analyzing the behavior of a function very close to a given value, from both the left and the right of the value—is a method that will be used often in other mathematics courses.

Locating the intercepts can help you determine if you correctly entered the parentheses when graphing a rational function on a graphing calculator.

Example 2 Intercepts of a Rational Graph

Find the intercepts of $f(x) = \dfrac{x^2 - x - 2}{x - 1}$.

Solution

The y-intercept is $f(0) = \dfrac{0^2 - 0 - 2}{0 - 1} = \dfrac{-2}{-1} = 2$.

The x-intercepts are solutions of $x^2 - x - 2 = 0$ that are not solutions of $x - 1 = 0$. Solutions of $x^2 - x - 2 = 0$ can be found by factoring.

$$x^2 - x - 2 = 0$$
$$(x + 1)(x - 2) = 0$$
$$x = -1 \quad \text{or} \quad x = 2.$$

Neither -1 nor 2 is a solution of $x - 1 = 0$, so both are x-intercepts of the graph of f, as shown in Figure 4.4-1.

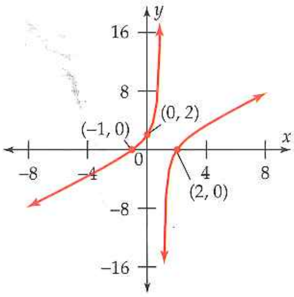


Figure 4.4-1

Continuity

There are breaks in the graph of a rational function wherever the function is not defined, that is, at the zeros of the denominator. Except for breaks, the graph is a continuous unbroken curve. Additionally, the graph has no sharp corners.

Vertical Asymptotes

Unlike polynomial functions, a rational function has breaks in its graph at all points where the function is not defined. Vertical asymptotes occur at every number that is a zero of the denominator but not of the numerator. The key to understanding the behavior of a rational function near these asymptotes is a fact from arithmetic.

The Big-Little Concept

If c is a number far from 0, then $\frac{1}{c}$ is a number close to 0.

If c is close to 0, then $\frac{1}{c}$ is far from 0.

In less precise, but more suggestive terms

$$\frac{1}{big} = little \qquad \text{and} \qquad \frac{1}{little} = big$$

For example, 5000 is big and $\frac{1}{5000}$ is little. Similarly, $\frac{-1}{1000}$ is little and $\frac{1}{\frac{-1}{1000}} = -1000$ is big. Note that even though -1000 is negative, it is far from zero and therefore is large in absolute value. The role played by the Big-Little Concept when graphing rational functions is illustrated in Example 3.

Example 3 A Rational Function Near a Vertical Asymptote

Without using a calculator, describe the graph of $f(x) = \frac{x+1}{2x-4}$ near $x = 2$. Then sketch the graph for values near $x = 2$.

Solution

The function is not defined at $x = 2$ because the denominator is 0 there. When x is greater than 2 but very close to 2,

- The numerator, $x + 1$, is very close to $2 + 1 = 3$.
- The denominator, $2x - 4$, is a positive number very close to $2(2) - 4 = 0$.

By the Big-Little Concept,

$$f(x) = \frac{x+1}{2x-4} = \frac{3}{little} = 3 \cdot \frac{1}{little} = 3(big) = very\ big$$

This fact can be confirmed by a table of values for $f(x)$ near $x = 2$ when $x = 2.01, 2.001, 2.0001$, etc., as shown in Figure 4.4-2a. In graphical terms, the points with x-coordinates slightly greater than 2 have very large y-coordinates, so the graph shoots upward just to the right of $x = 2$. That is,

f increases without bound as x approaches 2 from the right.

X	Y1
.01	150.5
.001	1500.5
.0001	15001
	ERROR
.9999	-15000
.999	-1500
.99	-149.5

=ERROR

Figure 4.4-2a

A similar analysis when x is less than 2 but very close to 2 shows that the numerator, $x + 1$, is very close to 3 and the denominator is negative and very close to 0. Using the Big-Little Concept, the quotient is a negative number far from 0. As x approaches 2 from values less than 2, the quotient becomes a larger and larger negative number. Therefore, the graph of f shoots downward just to left of $x = 2$. That is,

f decreases without bound as x approaches 2 from the left.

The portion of the graph of f near $x = 2$ is shown in Figure 4.4–2b.

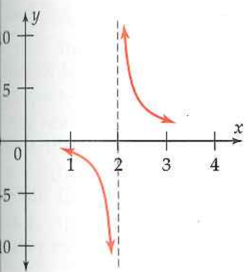


Figure 4.4-2b

The dashed vertical line in Figure 4.4-2b is included for easier visualization, but it is *not* part of the graph. Such a line is called a **vertical asymptote** of the graph. The graph approaches a vertical asymptote very closely, but never touches or crosses it because the function is not defined at that value of x.

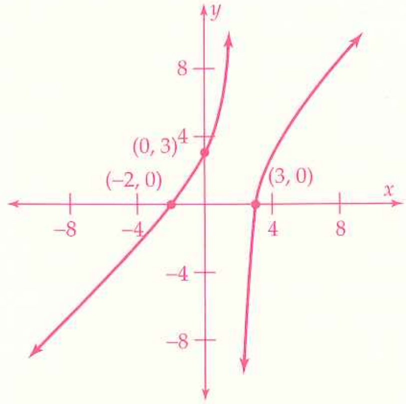
ADDITIONAL EXAMPLES

Example 2

Find the intercepts of $f(x) = \frac{x^2 - x - 6}{x - 2}$.

y-intercept: 3

x-intercepts: -2 and 3



Example 3

Without using a calculator, describe the graph of $f(x) = \frac{x+2}{3x-9}$ near $x = 3$. Then sketch the graph for values near $x = 3$.

The function is not defined for $x = 3$ because the denominator is 0 there. f increases without bound as x approaches 3 from the right. f decreases without bound as x approaches 3 from the left.

X	Y1
3.01	167
3.001	1667
3.0001	16667
3	ERROR
2.9999	-16666
2.999	-1666
2.99	-166.3

Y1=ERROR