4.3 Graphs of Polynomial Functions

Objectives

- Recognize the shape of basic polynomial functions
- Describe the graph of a polynomial function
- Identify properties of general polynomial functions: Continuity, End Behavior, Intercepts, Local Extrema, Points of Inflection
- Identify complete graphs of polynomial functions

The graph of a first-degree polynomial function is a straight line, as discussed in Section 1.4. The graph of a second-degree, or quadratic, polynomial function is a parabola, as discussed in Section 3.3. The emphasis in this section is on higher degree polynomial functions.

Basic Polynomial Shapes

The simplest polynomial functions are those of the form $f(x) = ax^n$, where a is a constant and n is a nonnegative integer. The graphs of polynomial functions of the form $f(x) = ax^n$, with $n \geq 2$, are of two basic types. The different types are determined by whether n is even or odd.

Polynomial Functions of Odd Degree

When the degree of a polynomial function in the form $f(x) = ax^n$, is odd, its graph has the basic form shown is Figures 4.3-1a and 4.3-1b. Notice that the graph shown in Figure 4.3-1b has the same shape as the graph shown in Figure 4.3-1a, but it is the reflection of the Figure 4.3-1a across either the x-axis or the y-axis.

$f(x) = ax^n$, n odd

$a > 0$

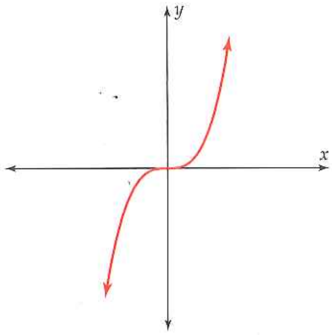


Figure 4.3-1a

$a < 0$

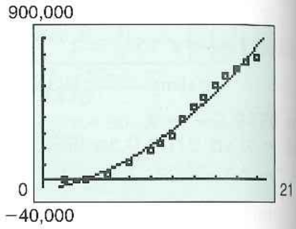


Figure 4.3-1b

Graphing Exploration

Graph each of the following functions of odd degree in the window with $-5 \leq x \leq 5$ and $-30 \leq y \leq 30$, and compare each shape with those shown in Figure 4.3-1a and 4.3-1b.

- $f(x) = 2x^3$
- $h(x) = -x^3$
- $g(x) = 0.01x^5$
- $k(x) = -2x^7$

Section 4.3 Graphs of Polynomial Functions

Teaching Notes

The graphs of polynomial functions of the form $f(x) = ax^n$ depend on two parameters, a and n. To show how a affects a graph, have students graph the following functions on the same screen, making these observations:

$f(x) = x^3$, $g(x) = \frac{1}{4}x^3$, and $h(x) = 4x^3$

$|a| < 1$ causes a vertical compression, and $|a| > 1$ causes a vertical stretch (see page 179).

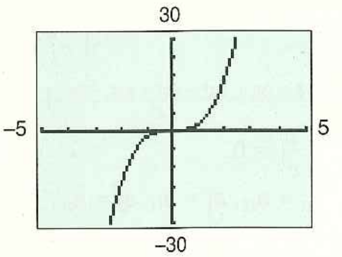
Repeat the activity to show how n affects a graph, using these functions:

$f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^{15}$

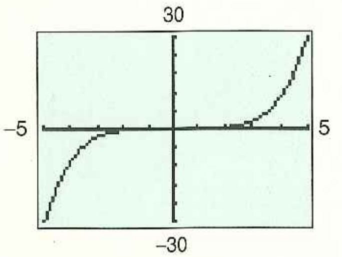
For $|x| < 1$, a greater value of n causes a vertical compression.

For $|x| > 1$, a greater value of n causes a vertical stretch.

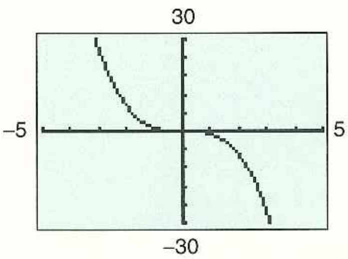
Solution to the **Graphing Exploration:**



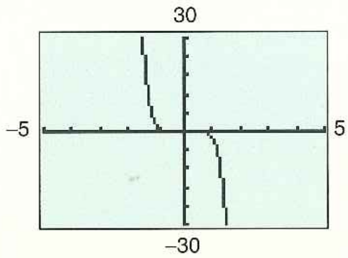
The graph of f is like the graph in Figure 4.3-1a.



The graph of g is like the graph in Figure 4.3-1a.



The graph of h is like the graph in Figure 4.3-1b.



The graph of k is like the graph in Figure 4.3-1b.

Polynomial Functions of Even Degree

When the degree of a polynomial function in the form $f(x) = ax^n$ is even, its graph has the form shown in Figures 4.3-2a or 4.3-2b. Again, the graph of $f(x) = ax^n$, when a is negative, is the reflection of Figure 4.3-2a across the x-axis.

$f(x) = ax^n$, n even

$a > 0$

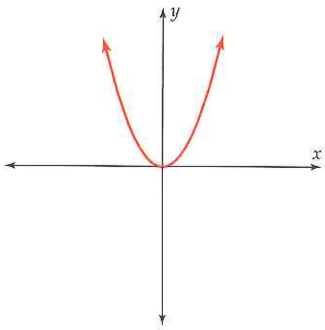


Figure 4.3-2a

$a < 0$

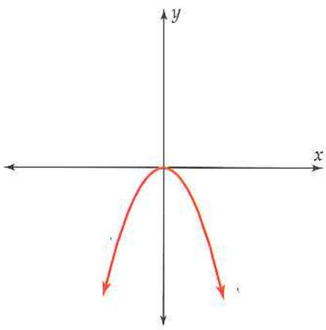


Figure 4.3-2b

Graphing Exploration

Graph each of the following functions of even degree in the window with $-5 \le x \le 5$ and $-30 \le y \le 30$, and compare each shape with those shown in Figure 4.3-2a or 4.3-2b.

- $f(x) = 2x^4$
- $h(x) = -2x^2$
- $g(x) = 6x^6$
- $k(x) = -3x^4$

Properties of General Polynomial Functions

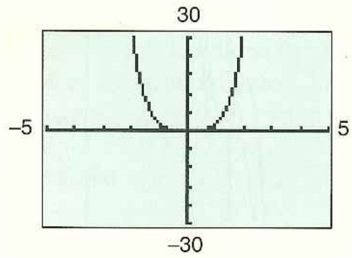
The graphs of other polynomial functions can vary considerably in shape. Understanding the properties that follow should assist you in interpreting graphs correctly and in determining when a graph of a polynomial function is complete.

Continuity

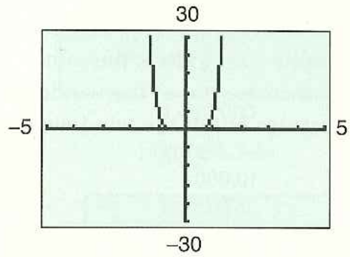
Every graph of a polynomial function is **continuous,** that is, it is an unbroken curve, with no jumps, gaps, or holes. Furthermore, graphs of polynomial functions have no sharp corners. Thus, *neither* of the graphs shown in Figure 4.3-3 represents a polynomial function. Note: some calculator graphs of polynomial functions may appear to have sharp corners; however, zooming in on the area in question will show a smooth curve.

Teaching Notes

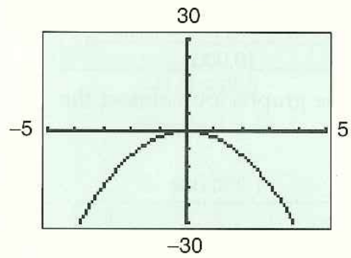
Solution to the **Graphing Exploration:**



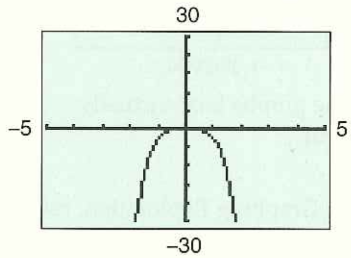
The graph of f is like the graph in Figure 4.3-2a.



The graph of g is like the graph in Figure 4.3-2a.



The graph of h is like the graph in Figure 4.3-2b.



The graph of k is like the graph in Figure 4.3-2b.

When discussing **continuity** remind students of the parent functions they studied (page 173). $f(x) = [x]$ and $f(x) = \frac{1}{x}$ are examples of functions that are *not* continuous.

ching Notes

olution to the **Graphing Exploration:**

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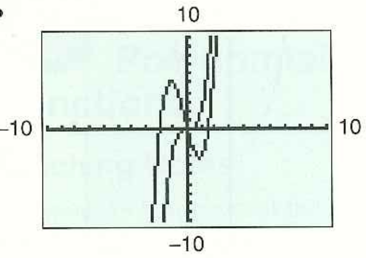

possible answer: In the standard viewing window, the graph of f has a local maximum and a local minimum; the graph of g does not. The graphs start to look the same as they increase out of the window and decrease out of the window.

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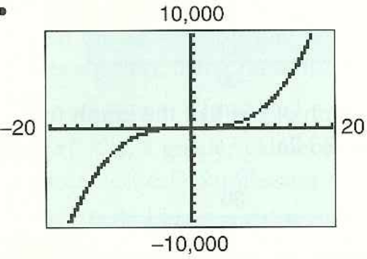
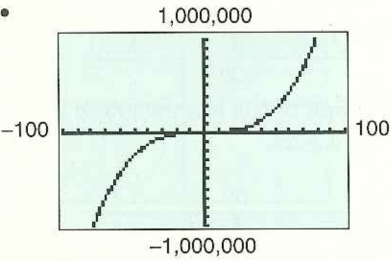


yes; the graphs look almost the same

•



yes; the graphs look virtually identical

After the Graphing Exploration, refer students to the table at the right. Ask them to describe the relationship among the entries in the last column, under the x-value 100. 2,009,400 is the sum of -600, 10,000, and 2,000,000. Emphasize that for values of x with large absolute values, the terms x^2 and $-6x$ in the polynomial $2x^3 + x^2 - 6x$ contribute relatively little to the function value.

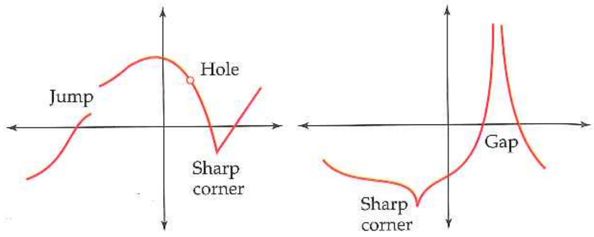


Figure 4.3-3

End Behavior

The shape of a polynomial graph at the far left and far right of the coordinate plane, that is, when $|x|$ is large, is called the **end behavior** of the graph. End behavior of graphs of functions of the form $f(x) = ax^n$ have common characteristics when n is odd and when n is even. The Graphing Exploration below asks you to find a generalization about the end behavior of polynomial functions of odd degree.

Graphing Exploration

Consider the function $f(x) = 2x^3 + x^2 - 6x$ and the function determined by its leading term $g(x) = 2x^3$.

- In a standard viewing window, graph f and g. Describe how the graphs look different and how they look the same.
- In the viewing window $-20 \le x \le 20$ and $-10{,}000 \le y \le 10{,}000$, graph f and g. Do the graphs look almost the same?
- In the viewing window

$$-100 \le x \le 100 \text{ and } -1{,}000{,}000 \le y \le 1{,}000{,}000,$$

graph f and g. Do the graphs look virtually identical?

The reason that the answer to the last question is "yes" can be understood by observing which term contributes the most to the output value $f(x)$ when x is large, as shown in the following chart.

Values of Specific Terms of $f(x) = 2x^3 + x^2 - 6x$

x	-100	-50	70	100
$-6x$	600	300	-420	-600
x^2	10,000	2,500	4,900	10,000
$g(x) = 2x^3$	$-2{,}000{,}000$	$-250{,}000$	686,000	2,000,000
$f(x) = 2x^3 + x^2 - 6x$	$-1{,}989{,}400$	$-247{,}200$	690,480	2,009,400

The chart shows that when $|x|$ is large, the terms x^2 and $-6x$ are insignificant compared with $2x^3$, and they play a very minor role in determining the end behavior of $f(x)$. Hence, the values of $f(x)$ and $g(x)$ are relatively close for large values of x.

End Behavior of Polynomial Functions

When $|x|$ is large, the graph of a polynomial function closely resembles the graph of its highest degree term.

When a polynomial function has *odd* degree, one end of its graph shoots upward and the other end downward.

When a polynomial function has *even* degree, both ends of its graph shoot upward or both ends shoot downward.

Following are some illustrations of the facts listed in the preceding box. In Figures 4.3-4a–d, the graph of a polynomial function is shown on the left and the graph of its leading term is shown on the right. The end behavior of the graph of the polynomial is the same as the end behavior of the graph of the leading term. Note the degree of each set of graphs and whether the leading coefficient is positive or negative.

$f(x) = 2x^4 - 5x^2 + 2$ $\qquad$ $g(x) = 2x^4$

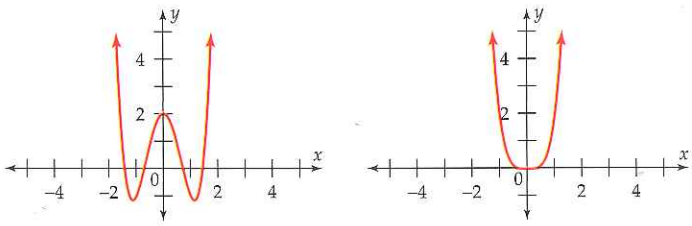


Figure 4.3-4a

$f(x) = -3x^6 + 5x^2 + 2$ $\qquad$ $g(x) = -3x^6$

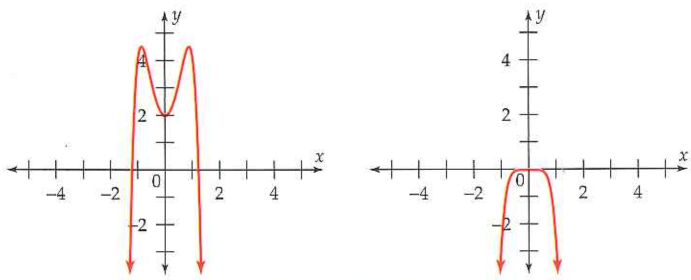


Figure 4.3-4b

Teaching Notes

When $|x|$ is large, the graph of the polynomial function resembles the graph of its highest term. Have students graph functions f and g together for each of Figures 4.3-4a through 4.3-4d in the viewing window $-100 \le x \le 100$, $-1{,}000{,}000 \le y \le 1{,}000{,}000$:

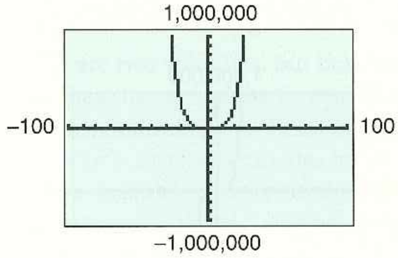


Figure 4.3-4a

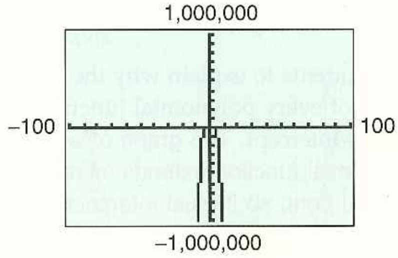


Figure 4.3-4b

continued from page 263

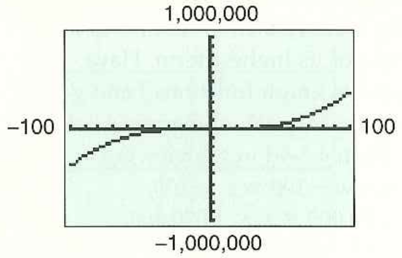


Figure 4.3-4c

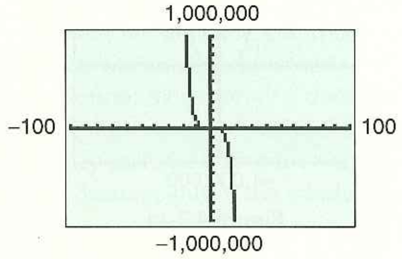


Figure 4.3-4d

Ask students to explain why the graph of every polynomial function has a y-intercept. The graph of every polynomial function extends infinitely left and right, so it must intersect the y-axis.

Ask students to explain why the graph of every polynomial function has *only one* y-intercept. The graph of every function must pass the vertical line test. That is, there cannot be more than one point of the graph on the y-axis.

Ask students to use the polynomial function form $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ to explain why the y-intercept is a_0. The y-intercept is the value of the function at $x = 0$.

$$f(0) = a_n(0)^n + a_{n-1}(0)^{n-1} + \cdots + a_1(0) + a_0 = a_0$$

Ask students which of the following functions *do not* have an x-intercept, and why:

$f(x) = x^2 - 9$ x-intercepts at -3 and 3

$f(x) = x^2 + 9$ no x-intercept because f has no real zeros

$f(x) = x^4$ x-intercept at zero

$f(x) = -x^4 - 16$ no x-intercept because f has no real zeros

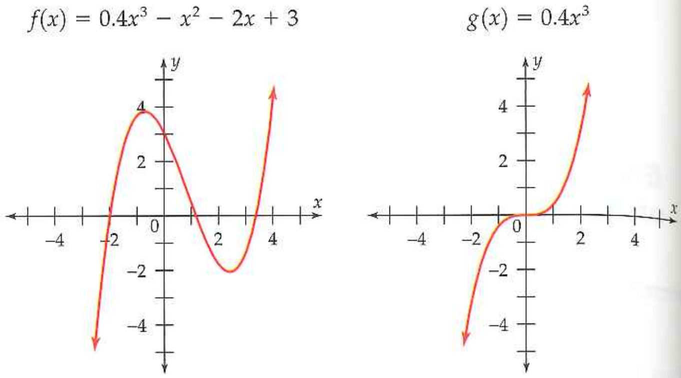


Figure 4.3-4c

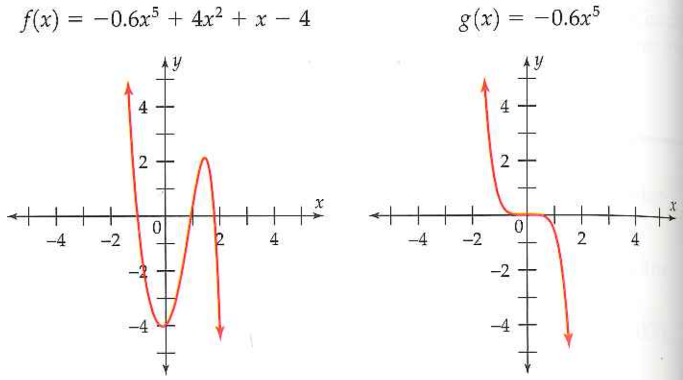


Figure 4.3-4d

Intercepts

Consider a polynomial function written in polynomial form.

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$$

- The y-intercept of the graph of f is the constant term, a_0.
- The x-intercepts of the graph of f are the real zeros of f.

The graph of every polynomial function has exactly one y-intercept, and because a polynomial of degree n has at most n distinct zeros, the number of x-intercepts is limited.

Intercepts

The graph of a polynomial function of degree n

- **has one y-intercept, which is equal to the constant term.**
- **has at most n x-intercepts.**

That is, the number of x-intercepts can be no greater than the degree of the polynomial function.

Multiplicity

There is another connection between zeros and graphs. If $x - r$ is a factor that occurs m times in the complete factorization of a polynomial expression, then r is called a zero with **multiplicity** m of the related polynomial function.

For example, -3, -1 and 1 are zeros of $f(x) = (x + 3)^2(x + 1)(x - 1)^3$. The multiplicity of each zero is shown in the following chart.

Zero	−3	−1	1
Multiplicity	2	1	3

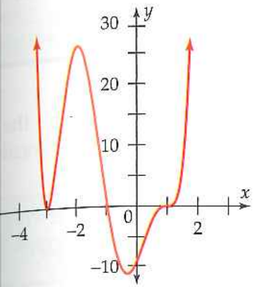


Figure 4.3-5

Observe in Figure 4.3-5 that the graph of f does not cross the x-axis at -3, a zero of *even* multiplicity, but does cross the x-axis at -1 and 1, zeros of *odd* multiplicity.

Multiplicity and Graphs

Let c be a zero of multiplicity k of a polynomial f.

- **If k is odd, the graph of f crosses the x-axis at c.**
- **If k is even, the graph of f touches, but does not cross, the x-axis at c.**

Example 1 Multiplicity of Zeros

Find all zeros of $f(x) = (x + 1)^2(x - 2)(x - 3)^3$. State the multiplicity of each zero, and state whether the graph of f touches or crosses the x-axis at each corresponding x-intercept.

Solution

The following chart lists the zeros of f, the multiplicity of each, and whether the graph touches or crosses the x-axis at the corresponding x-intercept.

Zero	Multiplicity	x-axis
−1	2	touches
2	1	crosses
3	3	crosses

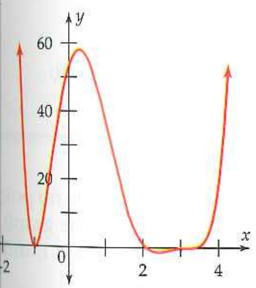


Figure 4.3-6

The graph of f, shown in Figure 4.3-6, verifies that the graph touches but does not cross the x-axis at -1 and crosses the x-axis at 2 and 3. ■

Teaching Notes

To introduce the concept of multiplicity, you may want to use a function such as $f(x) = x^2 + 4x + 4$ which can be written $f(x) = (x + 2)(x + 2)$.

Find the zeros:

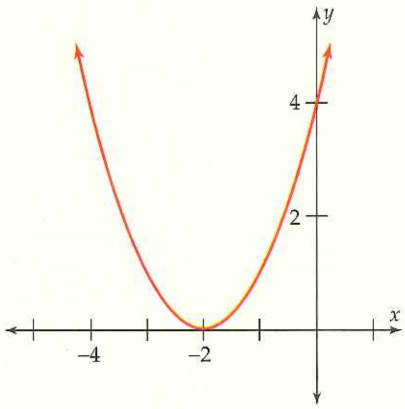
$(x + 2)(x + 2) = 0$

$x + 2 = 0$ or $x + 2 = 0$

$x = -2$ or $x = -2$

There are two solutions, but both are -2. Thus the zero of the function f is -2 with multiplicity 2. Point out that $f(x) = (x + 2)(x + 2)$ can also be written as $f(x) = (x + 2)^2$ and that the multiplicity of the zero is given by the exponent.

Then graph the function to show that it touches, but does not cross, the x-axis.

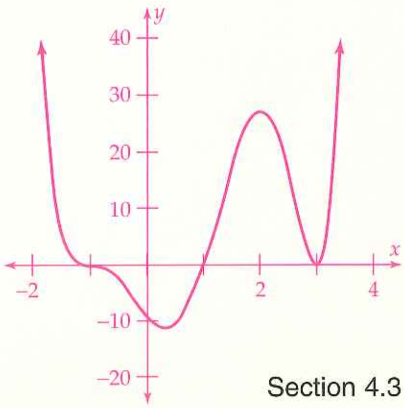


ADDITIONAL EXAMPLES

Example 1

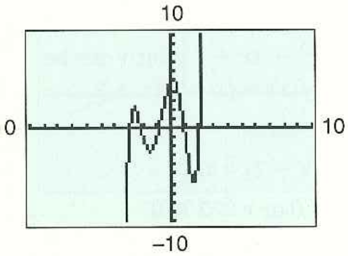
Find all zeros of $f(x) = (x + 1)^3(x - 1)(x - 3)^2$. State the multiplicity of each zero, and state whether the graph of f touches or crosses the x-axis at each corresponding x-intercept.

Zero	Multiplicity	x-axis
−1	3	crosses
1	1	crosses
3	2	touches

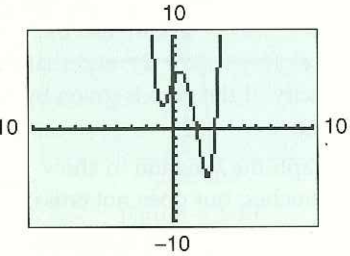


eaching Notes

lution to the **Graphing Exploration:**



here are 4 local extrema. The degree
f is 5.



here are 3 local extrema. The degree
g is 4.

ach polynomial function of degree n the Graphing Exploration has $n - 1$ cal extrema. Point out that the **umber of Local Extrema** for a olynomial function of degree n *at most* $n - 1$. Ask students to cate a function on page 263 that as *less than* $n - 1$ local extrema. $(x) = 2x^4$ or $g(x) = -3x^6$ or $x) = -3x^6 + 5x^2 + 2$

Iake sure students understand that if graph touches, but does not cross he x-axis, there is a local maximum r minimum at that point.

emind students (page 154) that a nction is concave up in a given terval if for any two points in the iven interval that lie on the curve, he segment that connects them is bove the curve. A function is concave own in a given interval if for any vo points in the given interval that e on the curve, the segment that onnects them is below the curve.

Iote to students that the second atement in **Number of Points of nflection** applies to any odd degree , where $n > 2$, so linear functions are ot included.

Local Extrema

The term **local extremum** (plural, extrema) refers to either a local maximum or a local minimum, that is, a point where the graph has a peak or a valley. Local extrema occur when the output values change from increasing to decreasing, or vice versa, as discussed in Section 3.2.

Graphing Exploration

- Graph $f(x) = 0.5x^5 + 1.5x^4 - 2.5x^3 - 7.5x^2 + 2x + 5$ in the standard viewing window. What is the total number of local extrema on the graph? What is the degree of f?
- Graph $g(x) = x^4 - 3x^3 - 2x^2 + 4x + 5$ in the standard viewing window. What is the total number of local extrema on the graph? What is the degree of $g(x)$?

The two polynomials graphed in the Exploration are illustrations of the following fact.

Number of Local Extrema

A polynomial function of degree n has at most $n - 1$ local extrema.

That is, that total number of peaks and valleys on the graph is at most one less than the degree of the function.

Points of Inflection

Recall from Section 3.2 that an inflection point occurs where the concavity of a graph of a function changes. The number of inflection points on the graph of a polynomial is governed by the degree of the function.

Number of Points of Inflection

- **The graph of a polynomial function of degree n, with $n \geq 2$, has at most $n - 2$ points of inflection.**
- **The graph of a polynomial function of odd degree, with $n > 2$, has at least one point of inflection.**

Thus, the graph of a quadratic function, which has degree 2, has no points of inflection because it can have at most $n - 2 = 2 - 2 = 0$. The graph of a cubic has exactly one point of inflection because it has at least 1 and at most $3 - 2 = 1$.

Technology Tip

Points of inflection may be found by using
NFLC in the TI-86/89
RAPH MATH menu.

Complete Graphs of Polynomial Function

By using the facts discussed in this section, you can often determine whether or not the graph of a polynomial function is complete, that is, shows all the important features.

Example 2 A Complete Graph of a Polynomial

Find a complete graph of $f(x) = x^4 + 10x^3 + 21x^2 - 40x - 80$.

Solution

Because the y-intercept is -80, graph f in the window with $-10 \le x \le 10$ and $-100 \le y \le 100$, as shown in Figure 4.3-7.

The three peaks and valleys shown are the only local extrema because a fourth-degree polynomial graph has at most three local extrema.

There cannot be more x-intercepts than the two shown because if the graph turned toward the x-axis farther to the right or farther to the left, there would be an additional peak, which is impossible.

Finally, the end behavior of the graph resembles the graph of $y = x^4$, the highest degree term.

Figure 4.3-7 includes all the important features of the graph and is therefore complete.

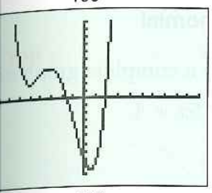


Figure 4.3-7

Example 3 A Complete Graph of a Polynomial

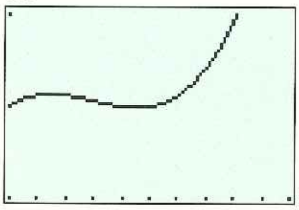
Find a complete graph of $f(x) = x^3 - 1.8x^2 + x + 2$.

Solution

The graph of f, shown in Figure 4.3-8a on the next page, is similar to the graph of its leading term $y = x^3$, but it does not appear to have any local extrema. However, if you use the trace feature on the flat portion of the graph to the right of the y-axis, you should see that the y-coordinates increase, then decrease, and then increase again.

Zoom in on the portion of the graph between 0 and 1, as shown in Figure 4.3-8b. Observe that the graph actually has two local extrema, one peak and one valley, which is the maximum possible number of local extrema for a cubic function. Figures 4.3-8a and 4.3-8b together provide a complete graph of f.

NOTE No polynomial graph of a function of degree $n > 1$ contains horizontal line segments like those shown in Figure 4.3-8a. Always investigate such segments by using trace or zoom-in to determine any hidden behavior.

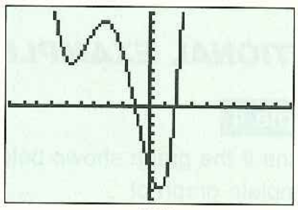


The two graphs provide a complete graph of f.

ADDITIONAL EXAMPLES

Example 2

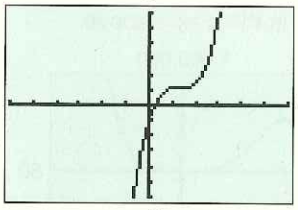
Find a complete graph of
$f(x) = x^4 + 11x^3 + 25x^2 - 50x - 70$.



The y-intercept is -70.
The three peaks and valleys shown are the only local extrema because a fourth-degree polynomial graph has at most three local extrema.
There cannot be more x-intercepts than the two shown because if the graph turned toward the x-axis farther to the right or farther to the left, there would be an additional peak, which is impossible.
The end behavior of the graph resembles the graph of $y = x^4$, the highest degree term.
The graph is complete.

Example 3

Find a complete graph of
$f(x) = x^3 - 3.9x^2 + 5x - 1$.



The graph of f is similar to the graph of its leading term, $y = x^3$, but does not appear to have any local extrema. Use the trace feature on the flat portion of the graph to show that the y-coordinates increase, decrease, and increase again. Zoom in on the portion of the graph between 1 and 2 to show that the graph has two local extrema, one peak and one valley, which is the maximum number of local extrema for a cubic function.

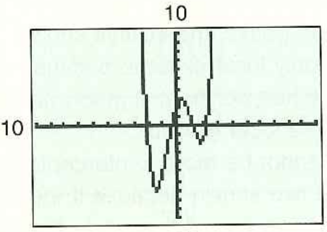
xample Notes

r **Example 4,** to help students see
hy both graphs are needed, point
ut that Figure 4.3-9b shows an
intercept that is not shown in
gure 4.3-9a.

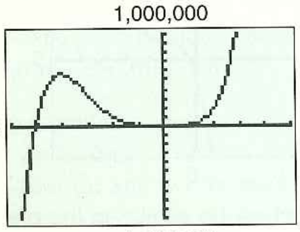
DDITIONAL EXAMPLES

xample 4

etermine if the graph shown below
a complete graph of
$x) = 0.02x^5 + x^4 - x^3 - 5x^2 + 4x + 2$.



his cannot be a complete graph
ecause when $|x|$ is large, the
raph must resemble the graph of
$(x) = 0.02x^5$, whose left end goes
ownward. The graph of f must turn
ownward and cross the x-axis
omewhere to the left of the origin.
o, the graph must have one more
eak where the graph turns
ownward, and must have another
-intercept. The graph below displays
he local maximum and x-intercept not
ncluded in the graph above.



oth graphs are necessary to show
all 5 x-intercepts and all 4 local
extrema of the function.

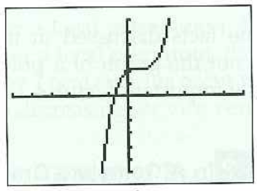


Figure 4.3-8a

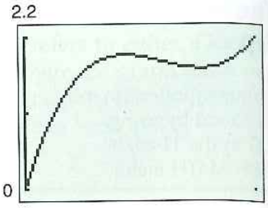


Figure 4.3-8b

Example 4 A Complete Graph of a Polynomial

Determine if the graph shown in Figure 4.3-9a is a complete graph of

$$f(x) = 0.01x^5 + x^4 - x^3 - 6x^2 + 5x + 4.$$

Solution

The graph shown in Figure 4.3-9a cannot be a complete graph because when $|x|$ is large, the graph of f must resemble the graph of $g(x) = 0.01x^5$, whose left end goes downward.

So the graph of f must turn downward and cross the x-axis somewhere to the left of the origin. Therefore, the graph must have one more peak, where the graph turns downward, and must have another x-intercept.

One additional peak and the ones shown in Figure 4.3-9a make a total of four, the maximum possible for a polynomial of degree 5. Similarly, the additional x-intercept makes a total of 5 x-intercepts. Because f has degree 5, there are no other x-intercepts.

A viewing window that includes the local maximum and the x-intercept shown in Figure 4.3-9b will not display the local extrema and x-intercepts shown in Figure 4.3-9a. Consequently, a complete graph of f requires both Figure 4.3-9a and Figure 4.3-9b to illustrate the important features of the graph.

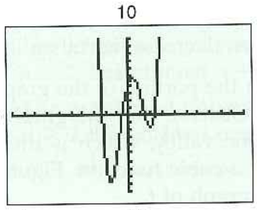


Figure 4.3-9a

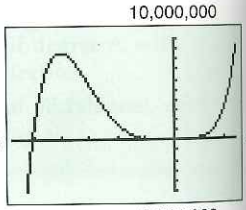


Figure 4.3-9b

The graphs shown in Examples 2–4 were known to be complete because they included the maximum possible number of local extrema. Many graphs, however, may have fewer than the maximum number of possible peaks and valleys. In such cases, use any available information and try several viewing windows to obtain the most complete graph.