55. Possible answer: $f(x) = (x - 1)(x - 7)(x + 4)$

56. Possible answer: $f(x) = (x - 1)^2(x + 1)$

57. Possible answer: $f(x) = (x - 1)(x - 2)^2(x - \pi)^3$

58. Possible answer: $f(x) = (x - 2)^5$

59. $f(x) = \frac{17}{100}(x - 5)(x - 8)x$

66. a. If $f(x) = x^n - c^n$, then $f(c) = c^n - c^n = 0$ and $x - c$ is a factor of f.

b. Let n be even, then $f(-c) = (-c)^n - c^n = c^n - c^n = 0$. Therefore, $x - (-c) = x + c$ is a factor of $f(x) = x^n - c^n$ if n is even.

67. a. If $n = 3$ and $c = 1$, then $x + 1 = x - (-1)$ is not a factor of $x^3 - 1$ since -1 is not a solution of $x^3 - 1 = 0$.

b. Since n is odd $(-c)^n = -c^n$ and hence $-c$ is a solution of $x^n + c^n = 0$. Thus, $x - (-c) = x + c$ is a factor of $x^n + c^n$ by the Factor Theorem.

In Exercises 55–58, find a polynomial with the given degree n, the given zeros, and no other zeros.

55. $n = 3$; zeros, 1, 7, -4

56. $n = 3$; zeros, 1, -1

57. $n = 6$; zeros 1, 2, π

58. $n = 5$; zero 2

59. Find a polynomial function f of degree 3 such that $f(10) = 17$ and the zeros of $f(x)$ are 0, 5, and 8.

60. Find a polynomial function g of degree 4 such that the zeros of g are 0, -1, 2, -3, and $g(3) = 288$.
$g(x) = 4x(x + 1)(x - 2)(x + 3)$

In Exercises 61–64, find a number k satisfying the given condition.

61. $x + 2$ is a factor of $x^3 + 3x^2 + kx - 2$.
$k = 1$

62. $x - 3$ is a factor of $x^4 - 5x^3 - kx^2 + 18x + 18$.
$k = 2$

63. $x - 1$ is a factor of $k^2x^4 - 2kx^2 + 1$.
$k = 1$

64. $x + 2$ is a factor of $x^3 - kx^2 + 3x + 7k$. $k = \frac{14}{3}$

65. Use the Factor Theorem to show that for every real number c, $x - c$ is not a factor of $x^4 + x^2 + 1$.
If $x - c$ were a factor of $x^4 + x^2 + 1$, then c would be a solution of $x^4 + x^2 + 1 = 0$, that is, c would satisfy $c^4 + c^2 = -1$. But $c^4 \geq 0$ and $c^2 \geq 0$, so that is impossible. Hence, $x - c$ is not a factor.

66. Let c be a real number and n a positive integer.
a. Show that $x - c$ is a factor of $x^n - c^n$.
b. If n is even, show that $x + c$ is a factor of $x^n - c^n$. [Remember: $x + c = x - (-c)$.]

67. a. If c is a real number and n an odd positive integer, give an example to show that $x + c$ may not be a factor of $x^n - c^n$.
b. If c and n are as in part **a,** show that $x + c$ is a factor of $x^n + c^n$.

68. *Critical Thinking* For what value of k is the difference quotient of $g(x) = kx^2 + 2x + 1$ equal to $7x + 2 + 3.5h$?
$k = 3.5$

69. *Critical Thinking* For what value of k is the difference quotient of $f(x) = x^2 + kx$ equal to $2x + 5 + h$?
$k = 5$

70. *Critical Thinking* When $x^3 + cx + 4$ is divided by $x + 2$, the remainder is 4. Find c.
$c = -4$

71. *Critical Thinking* If $x - d$ is a factor of $2x^3 - dx^2 + (1 - d^2)x + 5$, what is d?
$d = -5$

Section 4.2 Real Zeros

Math Background

It was shown by French mathematician Evariste Galois (1811–1832) that there are no formulas that can be used to solve general polynomial equations with degree greater than 4. The first proofs of this fact were by Niels Henrik Abel (1802–1829) of Norway and Paolo Ruffini (1765–1822) of Italy. Both proofs had some gaps and Ruffini's was not accepted at the time, but Abel's was. Galois determined which polynomials *could* be solved by a formula, which is a step beyond Abel and Ruffini.

4.2 Real Zeros

Objectives

- Find all rational zeros of a polynomial function
- Use the Factor Theorem
- Factor a polynomial completely
- Find lower and upper bounds of zeros

Finding the real zeros of a polynomial $f(x)$ is the same as solving the related polynomial equation, $f(x) = 0$. The zero of a first-degree polynomial, such as $5x - 3$, can always be found by solving the equation $5x - 3 = 0$. Similarly, the zeros of any second-degree polynomial can be found by using the quadratic formula, as discussed in Section 2.2. Although the zeros of higher degree polynomials can always be approximated graphically as in Section 2.1, it is better to find exact values, if possible.

Rational Zeros

When a polynomial has *integer* coefficients, all of its **rational zeros** (zeros that are rational numbers) can be found exactly by using the following test.

Teaching Notes

In this section, and in subsequent sections, it is important to pay attention to restrictions placed on coefficients of polynomial functions. In the discussion of **Rational Zeros,** the coefficients of the polynomials are *integers*.

The Rational Zero Test

> **If a rational number $\frac{r}{s}$ (written in lowest terms) is a zero of the polynomial function**
>
> $$f(x) = a_nx^n + \ \dots \ + a_1x + a_0$$
>
> **where the coefficients $a_n, \cdots, a_1$ are *integers* with $a_n \neq 0$ and $a_0 \neq 0$, then**
>
> - **r is a factor of the constant term a_0 and**
> - **s is a factor of the leading coefficient a_n.**

The test states that every rational zero of a polynomial function with integer coefficients must meet the conditions that

- the numerator is a factor of the constant term, and
- the denominator is a factor of the leading coefficient.

By finding all the numbers that satisfy these conditions, a list of *possible* rational zeros is produced. The polynomial must be evaluated at each number in the list to see if the number actually is a zero. Using a calculator can considerably shorten this testing process, as shown in the next example.

Example 1 The Rational Zeros of a Polynomial

Find the rational zeros of $f(x) = 2x^4 + x^3 - 17x^2 - 4x + 6$.

Solution

If $f(x)$ has a rational zero $\frac{r}{s}$, then by the Rational Zero Test r must be a factor of the constant term, 6. Therefore, r must be one of the integers $\pm1, \pm2, \pm3$, or ±6. Similarly, s must be a factor of the leading coefficient, 2. Therefore, s must be one of the integers ±1 or ±2. Consequently, the only *possibilities* for $\frac{r}{s}$, are $\frac{\pm1}{\pm1}, \frac{\pm2}{\pm1}, \frac{\pm3}{\pm1}, \frac{\pm6}{\pm1}, \frac{\pm1}{\pm2}, \frac{\pm2}{\pm2}, \frac{\pm3}{\pm2}, \frac{\pm6}{\pm2}$.

Eliminating duplications leaves a list of the only *possible* rational zeros.

$$1, -1, 2, -2, 3, -3, 6, -6, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}$$

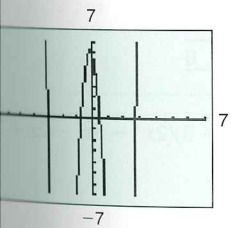


Figure 4.2-1a

Graph $f(x)$ in a viewing window that includes all of these numbers on the x-axis, such as $-7 \le x \le 7$ and $-7 \le y \le 7$. A complete graph is not necessary because only the x-intercepts are of interest.

The graph in Figure 4.2-1a shows that the only numbers in the list that could possibly be zeros are -3, $-\frac{1}{2}$, and $\frac{1}{2}$, so these are the only ones that need to be tested. The table feature can be used to evaluate $f(x)$ at

Example Notes

For **Example 1**, to show how **The Rational Zero Test** is used to find all possible rational zeros, write the following on the board:
factors of 6: $\pm1, \pm2, \pm3, \pm6$
factors of 2: $\pm1, \pm2$

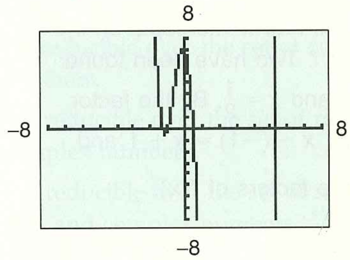
Now tell students to form all possible fractions, using factors of 6 as numerators and factors of 2 as denominators.

ADDITIONAL EXAMPLES

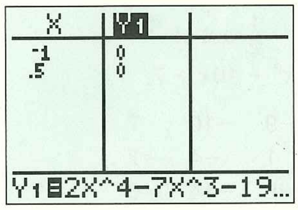
Example 1

Find the rational zeros of $f(x) = 2x^4 - 7x^3 - 19x^2 - 3x + 7$.

The only possible rational zeros are $1, -1, 7, -7, \frac{1}{2}, -\frac{1}{2}, \frac{7}{2}, -\frac{7}{2}$.
With viewing window $-8 \le x \le 8$ and $-8 \le y \le 8$, graph the function.



The graph shows that only the values -1 and $\frac{1}{2}$ could possibly be zeros of the function. Use the table feature:



-1 and $\frac{1}{2}$ are the rational zeros of f.
The two other zeros must be irrational.

Teaching Notes

Emphasize that to use **The Rational Zero Test**, the polynomial must have a *nonzero* constant term.

A polynomial with a *zero* constant term can be handled as follows:

$f(x) = a_nx^n + \cdots + a_1x + 0$
$f(x) = a_nx^n + \cdots + a_1x$
$f(x) = x(a_nx^{n-1} + \cdots + a_1)$

Now apply the Rational Zero Test to $a_nx^{n-1} + \cdots + a_1$ (See Exercise 5, page 258.)

Example Notes

For **Example 2**, remind students of the Factor Theorem (page 245): A polynomial function $f(x)$ has a linear factor $x - a$ if and only if $f(a) = 0$. You may also want to have students refer to the margin **NOTE** on page 253 showing $\frac{2 \pm \sqrt{12}}{2} = 1 \pm \sqrt{3}$. Have them verify that $x^2 - 2x - 2 = \left[x - (1 + \sqrt{3})\right]\left[x - (1 - \sqrt{3})\right]$. Point out that

$$f(x) = 2(x + 3)\left(x - \tfrac{1}{2}\right) \cdot \left[x - \left(\frac{2 + \sqrt{12}}{2}\right)\right]\left[x - \left(\frac{2 - \sqrt{12}}{2}\right)\right].$$

ADDITIONAL EXAMPLES

Example 2

Find all the real zeros of the function given in Additional Example 1.

$f(x) = 2x^4 - 7x^3 - 19x^2 - 3x + 7$

There are four x-intercepts on the graph of f. Two have been found: $x = -1$ and $x = \frac{1}{2}$. By the factor theorem, $x - (-1) = x + 1$ and $x - \frac{1}{2}$ are factors of $f(x)$.

Factor out $x + 1$.

-1	2	-7	-19	-3	7
		-2	9	10	-7
	2	-9	-10	7	0

$f(x) = (x + 1)(2x^3 - 9x^2 - 10x + 7)$

Factor $x - \frac{1}{2}$ out of $2x^3 - 9x^2 - 10x + 7$.

$\frac{1}{2}$	2	-9	-10	7
		1	-4	-7
	2	-8	-14	0

$f(x) = (x + 1)\left(x - \frac{1}{2}\right)(2x^2 - 8x - 14)$

$f(x) = 2(x + 1)\left(x - \frac{1}{2}\right)(x^2 - 4x - 7)$

The two remaining zeros of f are solutions of $x^2 - 4x - 7 = 0$.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-7)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{44}}{2} = \frac{4 \pm 2\sqrt{11}}{2} = 2 \pm \sqrt{11}$$

$f(x)$ has two rational zeros, -1 and $\frac{1}{2}$, and two irrational zeros, $2 + \sqrt{11}$ and $2 - \sqrt{11}$.

Technology Tip

The table setup screen is labeled TBLSET on the TI and RANG in the Casio TABLE menu.

these three numbers, as shown in Figure 4.2-1b, where the function is entered in the $Y =$ editor and the independent variable is set to Ask.

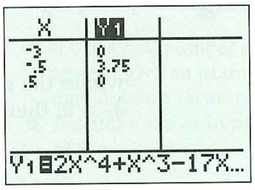


Figure 4.2-1b

The table shows that -3 and $\frac{1}{2}$ are the rational zeros of f and $-\frac{1}{2}$ is not a zero. Therefore, the two other zeros shown in Figure 4.2-1a cannot be rational numbers, that is, the two other zeros must be irrational.

Zeros and the Factor Theorem

Once some zeros of a polynomial have been found, the Factor Theorem can be used to factor the polynomial, which may lead to additional zeros.

Example 2 Finding All Real Zeros of a Polynomial

Find all the real zeros of the function given in Example 1.

$$f(x) = 2x^4 + x^3 - 17x^2 - 4x + 6$$

Solution

The graph of f, shown in Figure 4.2-1a, shows that there are four x-intercepts, and therefore, four real zeros. The rational zeros, -3 and $\frac{1}{2}$, were found in Example 1. By the Factor Theorem, $x - (-3) = x + 3$ and $x - \frac{1}{2}$ are factors of $f(x)$. The other factors can be found by using synthetic division twice. First, factor $x + 3$ out of $f(x)$.

-3	2	1	-17	-4	6
		-6	15	6	-6
	2	-5	-2	2	0

$$f(x) = (x + 3)(2x^3 - 5x^2 - 2x + 2)$$

Then factor $x - \frac{1}{2}$ out of $2x^3 - 5x^2 - 2x + 2$.

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -5 & -2 & 2 \\ & & 1 & -2 & -2 \\ \hline & 2 & -4 & -4 & \boxed{0} \end{array}$$

$$f(x) = (x + 3)\left(x - \frac{1}{2}\right)(2x^2 - 4x - 4)$$

Therefore, $2x^4 + x^3 - 17x^2 - 4x + 6 = (x + 3)\left(x - \frac{1}{2}\right)(2x^2 - 4x - 4)$

$$= 2(x + 3)\left(x - \frac{1}{2}\right)(x^2 - 2x - 2)$$

The two remaining zeros of f are the solutions of $x^2 - 2x - 2 = 0$, which can be found by using the quadratic formula.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} = \frac{2 \pm \sqrt{12}}{2}$$

NOTE Recall from algebra that $\frac{2 \pm \sqrt{12}}{2}$ can be simplified as follows: $\frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$.

Therefore, $f(x)$ has two rational zeros, -3 and $\frac{1}{2}$, and two irrational zeros, $\frac{2 + \sqrt{12}}{2}$ and $\frac{2 - \sqrt{12}}{2}$.

Irreducible and Completely Factored Polynomials

A polynomial that cannot be written as the product of polynomials of lesser degree is said to be **irreducible.** When a polynomial is written as the product of irreducible factors with real coefficients, it is said to be **completely factored over the set of real numbers.** All linear polynomials are irreducible, and some quadratic polynomials are irreducible over the set of real numbers.

Example 2 shows that $\frac{2 + \sqrt{12}}{2}$ and $\frac{2 - \sqrt{12}}{2}$ are zeros of $2x^2 - 4x - 4$. By the Factor Theorem $x - \left(\frac{2 + \sqrt{12}}{2}\right)$ and $x - \left(\frac{2 - \sqrt{12}}{2}\right)$ are factors. You can verify that

$$2x^2 - 4x - 4 = 2\left[x - \left(\frac{2 + \sqrt{12}}{2}\right)\right]\left[x - \left(\frac{2 - \sqrt{12}}{2}\right)\right].$$

Therefore, the original polynomial can be written as

$$f(x) = 2x^4 + x^3 - 17x^2 - 4x + 6$$
$$= 2(x + 3)\left(x - \frac{1}{2}\right)\left[x - \left(\frac{2 + \sqrt{12}}{2}\right)\right]\left[x - \left(\frac{2 - \sqrt{12}}{2}\right)\right].$$

Notice that the Factor Theorem applies to irrational zeros as well as to rational zeros. That is, because $\frac{2 + \sqrt{12}}{2}$ is a zero, $x - \left(\frac{2 + \sqrt{12}}{2}\right)$ is a factor.

Math Background

Whether a polynomial is reducible (factorable) depends on what set of numbers is used.

Consider the following polynomials:

$f(x) = x^2 + 9$

$g(x) = x^2 - 3$

$h(x) = x^2 - 9$

As shown below, f is reducible over the set of complex numbers (see Section 4.5), g is reducible over the set of real numbers, and h is reducible over the set of rational numbers.

$f(x) = (x + 3i)(x - 3i)$

$g(x) = (x + \sqrt{3})(x - \sqrt{3})$

$h(x) = (x + 3)(x - 3)$

Furthermore, because all rational numbers are real numbers and all real numbers are complex numbers, we can restate these facts as follows:

f is reducible over the set of complex numbers.

g is reducible over the set of real and complex numbers.

h is reducible over the set of rational, real, and complex numbers.

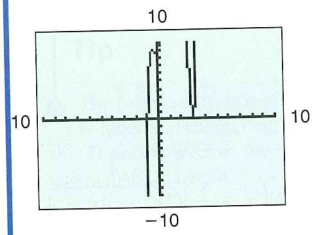


Figure 4.2-2

Example 3 **Factoring a Polynomial Completely**

Factor $f(x) = 2x^5 - 10x^4 + 7x^3 + 13x^2 + 3x + 9$ completely.

Solution

Begin by finding as many rational zeros as possible. By the Rational Zero Test, every rational zero is of the form $\frac{r}{s}$, where $r = \pm1, \ \pm3$, or ±9 and $s = \pm1$ or ±2. Thus, the possible rational zeros are

$$\pm1, \pm3, \pm9, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{9}{2}$$

The graph of f shows that the only possible zeros are -1 and 3. It is easily verified that both numbers *are* zeros of $f(x)$. Consequently, $x - (-1) = x + 1$ and $x - 3$ are factors of $f(x)$ by the Factor Theorem. Division shows other factors.

$$\begin{aligned} 2x^5 - 10x^4 + 7x^3 + 13x^2 + 3x + 9 &= (x + 1)(2x^4 - 12x^3 + 19x^2 - 6x + 9) \\ &= (x + 1)(x - 3)(2x^3 - 6x^2 + x - 3) \end{aligned}$$

The other zeros of f are the zeros of $g(x) = 2x^3 - 6x^2 + x - 3$.

Because every zero of $g(x)$ is also a zero of $f(x)$, and the only rational zeros of f are -1 and 3, check if either is a zero of g by using substitution.

$$g(-1) = 2(-1)^3 - 6(-1)^2 + (-1) - 3 = -12$$
$$g(3) = 2(3)^3 - 6(3)^2 + (3) - 3 = 0$$

So 3 is a zero of g, but -1 is not. By the Factor Theorem, $x - 3$ is a factor of $g(x)$. Division shows that

$$\begin{aligned} f(x) &= (x + 1)(x - 3)(2x^3 - 6x^2 + x - 3) \\ &= (x + 1)(x - 3)(x - 3)(2x^2 + 1) \end{aligned}$$

Because $2x^2 + 1$ has no real zeros, it cannot be factored further. So $f(x)$ is completely factored in the last statement above.

Bounds

In some cases, special techniques may be needed to guarantee that all zeros of a polynomial are located.

Example 4 **Finding All Real Zeros of a Polynomial**

Show that all the real zeros of $g(x) = x^5 - 2x^4 - x^3 + 3x + 1$ lie between -1 and 3, and find all the real zeros of $g(x)$.

Solution

First show that g has no zero larger than 3, as follows. Use synthetic division to divide $g(x)$ by $x - 3$.

Example Notes

In **Example 3**, $f(x) = (x + 1)(x - 3)(2x^3 - 6x^2 + x - 3)$. By letting $g(x) = (2x^3 - 6x^2 + x - 3)$, we can say every zero of g is a zero of f. By the Rational Zero Test, the possible rational zeros of $g(x) = 2x^3 - 6x^2 + x - 3$ are $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm1$, and ±3. However, since it is already established that the only rational zeros of f are -1 and 3, we only need to check -1 and 3 when finding the zeros of g.

Note that one of the zeros of g is 3, so it turns out that $x - 3$ is a repeated factor of f.

Math Background

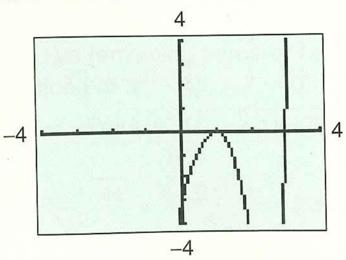
From Figure 4.2-2, it appears that 3 is a zero of f (which would make $x - 3$ a factor of $f(x)$). We find in **Example 3** that $x - 3$ is a *repeated* factor of $f(x)$. Students will learn in Section 4.3 (page 265) that when a factor is repeated an even number of times, the graph of the function touches, but does not cross, the x-axis at the corresponding zero.

ADDITIONAL EXAMPLES

Example 3

Factor $f(x) = x^5 - 5x^4 + 8x^3 - 8x^2 + 7x - 3$ completely.

Using the Rational Zero Test, the possible rational zeros for f are $\pm1, \pm3$. The graph of the function f shows that the only possible zeros are 1 and 3.



Substitution shows that both 1 and 3 are zeros of $f(x)$, and division shows that $f(x) = (x - 1)(x - 3)(x^3 - x^2 + x - 1)$. The other zeros of f are zeros of $x^3 - x^2 + x - 1$. Checking 1 reveals that 1 is a zero again. The factored form of f is $f(x) = (x - 1)(x - 1)(x - 3)(x^2 + 1)$. The function cannot be factored further, since $x^2 + 1$ has no real zeros.