as linear factors. Because its leading term is x^4, $(x - a)(x - b)(x - c)(x - d)$ has degree 4. Since $f(x)$ must have all four factors, its degree must be at least 4. In particular, this means that no polynomial of degree 3 can have four or more zeros. A similar argument holds in the general case.

Number of Zeros

A polynomial of degree n has at most n distinct real zeros.

Example 9 **Maximum Number of Distinct Real Zeros**

State the maximum number of distinct real zeros of f.

$$f(x) = 18x^4 - 51x^3 - 187x^2 - 56x + 80$$

Solution

The degree of f is 4. Therefore, the maximum number of distinct real zeros of f is 4.

Exercises 4.1

In Exercises 1–8, determine whether the given algebraic expression is a polynomial. If it is, list its leading coefficient, constant term, and degree.

1. $1 + x^3$ **2.** -7 **3.** $(x - 1)(x^2 + 1)$

4. $7^x + 2x + 1$ **5.** $(x + \sqrt{3})(x - \sqrt{3})$

6. $4x^2 + 3\sqrt{x} + 5$ **7.** $\frac{7}{x^2} + \frac{5}{x} - 15$

8. $(x - 1)^k$ (where k is a fixed positive integer)

In Exercises 9–16, use synthetic division to find the quotient and remainder.

9. $(3x^4 - 8x^3 + 9x + 5) \div (x - 2)$
$3x^3 - 2x^2 - 4x + 1; 7$

10. $(4x^3 - 3x^2 + x + 7) \div (x - 2)$
$4x^2 + 5x + 11; 29$

11. $(2x^4 + 5x^3 - 2x - 8) \div (x + 3)$
$2x^3 - x^2 + 3x - 11; 25$

12. $(3x^3 - 2x^2 - 8) \div (x + 5)$
$3x^2 - 17x + 85; -433$

13. $(5x^4 - 3x^2 - 4x + 6) \div (x - 7)$
$5x^3 + 35x^2 + 242x + 1690; 11{,}836$

14. $(3x^4 - 2x^3 + 7x - 4) \div (x - 3)$
$3x^3 + 7x^2 + 21x + 70; 206$

15. $(x^4 - 6x^3 + 4x^2 + 2x - 7) \div (x - 2)$
$x^3 - 4x^2 - 4x - 6; -19$

16. $(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1) \div (x + 3)$
$x^5 - 4x^4 + 13x^3 - 40x^2 + 121x - 364; 1093$

In Exercises 17–22, state the quotient and remainder when the first polynomial is divided by the second. Check your division by calculating (Divisor)(Quotient) + Remainder

17. $3x^4 + 2x^2 - 6x + 1$; $x + 1$
$3x^3 - 3x^2 + 5x - 11; 12$

18. $x^5 - x^3 + x - 5$; $x - 2$
$x^4 + 2x^3 + 3x^2 + 6x + 13; 21$

19. $x^5 + 2x^4 - 6x^3 + x^2 - 5x + 1$; $x^3 + 1$
$x^2 + 2x - 6; -7x + 7$

20. $3x^4 - 3x^3 - 11x^2 + 6x - 1$; $x^3 + x^2 - 2$
$3x - 6; -5x^2 + 12x - 13$

21. $5x^4 + 5x^2 + 5$; $x^2 - x + 1$
$5x^2 + 5x + 5; 0$

22. $x^5 - 1$; $x - 1$
$x^4 + x^3 + x^2 + x + 1; 0$

In Exercises 23–26, determine whether the first polynomial is a factor of the second.

23. $x^2 + 3x - 1$; $x^3 + 2x^2 - 5x - 6$ **No**

24. $x^2 + 9$; $x^5 + x^4 - 81x - 81$ **Yes**

Math Background

A polynomial of degree n has exactly n zeros (see page 308). However, some zeros may not be distinct, and some may be complex numbers. Consider the fourth-degree polynomial $f(x) = (x - 5)(x - 5)(x + 3i)(x - 3i)$. It has four zeros, which are $x = 5$, $x = 5$, $x = -3i$, and $x = 3i$. 5 is a zero of multiplicity 2, and $3i$ and $-3i$ are complex zeros. This polynomial has only one distinct real zero, namely 5.

ADDITIONAL EXAMPLES

Example 9

State the maximum number of distinct real zeros of f.

$f(x) = 6x^5 + 2x^3 - x + 7$

The degree of f is 5. Thus, the maximum number of distinct real zeros of f is 5.

Exercises 4.1

ANSWERS

1. Polynomial of degree 3; leading coefficient 1; constant term 1
2. Polynomial of degree 0; leading coefficient −7; constant term −7
3. Polynomial of degree 3; leading coefficient 1; constant term −1
4. Not a polynomial
5. Polynomial of degree 2; leading coefficient 1; constant term −3
6. Not a polynomial
7. Not a polynomial
8. Polynomial of degree k; leading coefficient 1; constant term either 1 or −1, depending on whether the degree k is even or odd

25. $x^2 + 3x - 1$; $x^4 + 3x^3 - 2x^2 - 3x + 1$
Yes

26. $x^2 - 5x + 7$; $x^3 - 3x^2 - 3x + 9$
No

In Exercises 27–30, determine which of the given numbers are zeros of the given polynomial.

27. $2, 3, 0, -1$; $g(x) = x^4 + 6x^3 - x^2 - 30x$
0, 2

28. $1, \frac{1}{2}, 2, -\frac{1}{2}, 3$; $f(x) = 6x^2 + x - 1$ $-\frac{1}{2}$

29. $2\sqrt{2}, \sqrt{2}, -\sqrt{2}, 1, -1$; $h(x) = x^3 + x^2 - 8x - 8$
$2\sqrt{2}, -1$

30. $\sqrt{3}, -\sqrt{3}, 1, -1$; $k(x) = 8x^3 - 12x^2 - 6x + 9$
None

In Exercises 31–40, find the remainder when $f(x)$ is divided by $g(x)$, without using division.

31. $f(x) = x^{10} + x^8$; $g(x) = x - 1$
2

32. $f(x) = x^6 - 10$; $g(x) = x - 2$
54

33. $f(x) = 3x^4 - 6x^3 + 2x - 1$; $g(x) = x + 1$
6

34. $f(x) = x^5 - 3x^2 + 2x - 1$; $g(x) = x - 2$
23

35. $f(x) = x^3 - 2x^2 + 5x - 4$; $g(x) = x + 2$
−30

36. $f(x) = 10x^{75} - 8x^{65} + 6x^{45} + 4x^{32} - 2x^{15} + 5$; $g(x) = x - 1$
15

37. $f(x) = 2x^5 - 3x^4 + x^3 - 2x^2 + x - 8$; $g(x) = x - 10$
170,802

38. $f(x) = x^3 + 8x^2 - 29x + 44$; $g(x) = x + 11$
0

39. $f(x) = 2x^5 - 3x^4 + 2x^3 - 8x - 8$; $g(x) = x - 20$
5,935,832

40. $f(x) = x^5 - 10x^4 + 20x^3 - 5x - 95$; $g(x) = x + 10$
−220,045

In Exercises 41–46, use the Factor Theorem to determine whether $h(x)$ is a factor of $f(x)$.

41. $h(x) = x - 1$; $f(x) = x^5 + 1$
No

42. $h(x) = x - \frac{1}{2}$; $f(x) = 2x^4 + x^3 + x - \frac{3}{4}$
Yes

43. $h(x) = x + 2$; $f(x) = x^3 - 3x^2 - 4x - 12$
No

44. $h(x) = x + 1$; $f(x) = x^3 - 4x^2 + 3x + 8$
Yes

45. $h(x) = x - 1$; $f(x) = 14x^{99} - 65x^{56} + 51$
Yes

46. $h(x) = x - 2$; $f(x) = x^3 + x^2 - 4x + 4$
No

In Exercises 47–50, use the Factor Theorem and a calculator to factor the polynomial, as in Example 7.

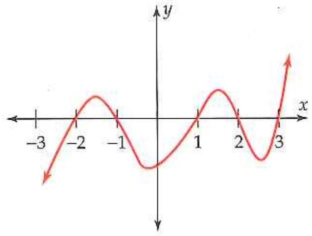
47. $f(x) = 6x^3 - 7x^2 - 89x + 140$
$(x + 4)(2x - 7)(3x - 5)$

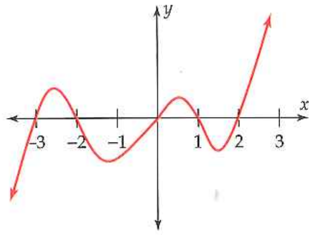
48. $g(x) = x^3 - 5x^2 - 5x - 6$
$(x - 6)(x^2 + x + 1)$

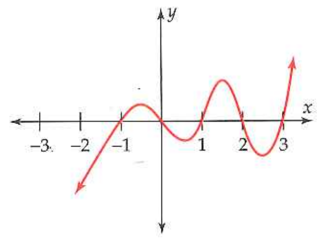
49. $h(x) = 4x^4 + 4x^3 - 35x^2 - 36x - 9$
$(x - 3)(x + 3)(2x + 1)^2$

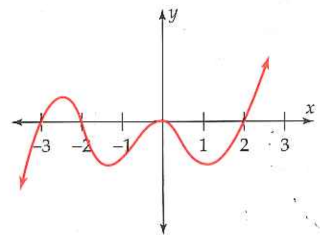
50. $f(x) = x^5 - 5x^4 - 5x^3 + 25x^2 + 6x - 30$
$(x - 5)(x^2 - 3)(x^2 - 2)$

In Exercises 51–54, each graph is of a polynomial function $f(x)$ of degree 5 whose leading coefficient is 1, but the graph is not drawn to scale. Use the Factor Theorem to find the polynomial. *Hint:* What are the zeros of $f(x)$? What does the Factor Theorem tell you?

51.


52.


53.


54.


51. $f(x) = (x + 2)(x + 1) \cdot (x - 1)(x - 2)(x - 3)$
$= x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12$

52. $f(x) = x(x + 3)(x + 2)(x - 1) \cdot (x - 2)$
$= x^5 + 2x^4 - 7x^3 - 8x^2 + 12x$

53. $f(x) = x(x + 1)(x - 1) \cdot (x - 2)(x - 3)$
$= x^5 - 5x^4 + 5x^3 + 5x^2 - 6x$

54. $f(x) = x^2(x + 3)(x + 2)(x - 2)$
$= x^5 + 3x^4 - 4x^3 - 12x^2$

5. Possible answer:
$f(x) = (x - 1)(x - 7)(x + 4)$

6. Possible answer:
$f(x) = (x - 1)^2(x + 1)$

7. Possible answer:
$f(x) = (x - 1)(x - 2)^2(x - \pi)^3$

8. Possible answer: $f(x) = (x - 2)^5$

9. $f(x) = \frac{17}{100}(x - 5)(x - 8)x$

6. a. If $f(x) = x^n - c^n$, then $f(c) = c^n - c^n = 0$ and $x - c$ is a factor of f.
b. Let n be even, then $f(-c) = (-c)^n - c^n = c^n - c^n = 0$. Therefore, $x - (-c) = x + c$ is a factor of $f(x) = x^n - c^n$ if n is even.

7. a. If $n = 3$ and $c = 1$, then $x + 1 = x - (-1)$ is not a factor of $x^3 - 1$ since -1 is not a solution of $x^3 - 1 = 0$.
b. Since n is odd $(-c)^n = -c^n$ and hence $-c$ is a solution of $x^n + c^n = 0$. Thus, $x - (-c) = x + c$ is a factor of $x^n + c^n$ by the Factor Theorem.

Section 4.2 Real Zeros

Math Background

was shown by French athematician Evariste Galois 811–1832) that there are no formulas hat can be used to solve general olynomial equations with degree reater than 4. The first proofs of this act were by Niels Henrik Abel 802–1829) of Norway and Paolo uffini (1765–1822) of Italy. Both roofs had some gaps and Ruffini's as not accepted at the time, but bel's was. Galois determined which olynomials *could* be solved by a rmula, which is a step beyond Abel nd Ruffini.

In Exercises 55–58, find a polynomial with the given degree n, the given zeros, and no other zeros.

55. $n = 3$; zeros, $1, 7, -4$

56. $n = 3$; zeros, $1, -1$

57. $n = 6$; zeros $1, 2, \pi$

58. $n = 5$; zero 2

59. Find a polynomial function f of degree 3 such that $f(10) = 17$ and the zeros of $f(x)$ are 0, 5, and 8.

60. Find a polynomial function g of degree 4 such that the zeros of g are $0, -1, 2, -3$, and $g(3) = 288$.
$g(x) = 4x(x + 1)(x - 2)(x + 3)$

In Exercises 61–64, find a number k satisfying the given condition.

61. $x + 2$ is a factor of $x^3 + 3x^2 + kx - 2$.
$k = 1$

62. $x - 3$ is a factor of $x^4 - 5x^3 - kx^2 + 18x + 18$.
$k = 2$

63. $x - 1$ is a factor of $k^2x^4 - 2kx^2 + 1$.
$k = 1$

64. $x + 2$ is a factor of $x^3 - kx^2 + 3x + 7k$. $k = \frac{14}{3}$

65. Use the Factor Theorem to show that for every real number c, $x - c$ is not a factor of $x^4 + x^2 + 1$.
If $x - c$ were a factor of $x^4 + x^2 + 1$, then c would be a solution of $x^4 + x^2 + 1 = 0$, that is, c would satisfy $c^4 + c^2 = -1$. But $c^4 \geq 0$ and $c^2 \geq 0$, so that is impossible. Hence, $x - c$ is not a factor.

66. Let c be a real number and n a positive integer.
a. Show that $x - c$ is a factor of $x^n - c^n$.
b. If n is even, show that $x + c$ is a factor of $x^n - c^n$. [Remember: $x + c = x - (-c)$.]

67. a. If c is a real number and n an odd positive integer, give an example to show that $x + c$ may not be a factor of $x^n - c^n$.
b. If c and n are as in part **a**, show that $x + c$ is a factor of $x^n + c^n$.

68. *Critical Thinking* For what value of k is the difference quotient of $g(x) = kx^2 + 2x + 1$ equal to $7x + 2 + 3.5h$?
$k = 3.5$

69. *Critical Thinking* For what value of k is the difference quotient of $f(x) = x^2 + kx$ equal to $2x + 5 + h$?
$k = 5$

70. *Critical Thinking* When $x^3 + cx + 4$ is divided by $x + 2$, the remainder is 4. Find c.
$c = -4$

71. *Critical Thinking* If $x - d$ is a factor of $2x^3 - dx^2 + (1 - d^2)x + 5$, what is d?
$d = -5$

4.2 Real Zeros

Objectives

- Find all rational zeros of a polynomial function
- Use the Factor Theorem
- Factor a polynomial completely
- Find lower and upper bounds of zeros

Finding the real zeros of a polynomial $f(x)$ is the same as solving the related polynomial equation, $f(x) = 0$. The zero of a first-degree polynomial, such as $5x - 3$, can always be found by solving the equation $5x - 3 = 0$. Similarly, the zeros of any second-degree polynomial can be found by using the quadratic formula, as discussed in Section 2.2. Although the zeros of higher degree polynomials can always be approximated graphically as in Section 2.1, it is better to find exact values, if possible.

Rational Zeros

When a polynomial has *integer* coefficients, all of its **rational zeros** (zeros that are rational numbers) can be found exactly by using the following test.

Teaching Notes

In this section, and in subsequent sections, it is important to pay attention to restrictions placed on coefficients of polynomial functions. In the discussion of Rational Zeros, the coefficients of the polynomials are *integers*.